Uniform Integrability

A collection of random variables $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ is called **uniformly integrable** (UI) if

$$\lim_{a \to \infty} \sup_{X \in \Lambda} \mathbb{E}[\,|X| : |X| \geq a\,] = 0.$$

I.e. tail expectations are <u>uniformly</u> small.

Eg. If Λ has a dominating L^1 function, $|X| \leq g \in L^1 \quad \forall X \in \Lambda$,

$$\sup_{X \in \Lambda} \mathbb{E}[\,\underbrace{|X|}_{\leq g}\, \underbrace{\mathbb{1}_{|X| \geq a}}_{\leq \mathbb{1}_{g \geq a}}\,] \qquad \{|X| \geq a\} \subseteq \{g \geq a\}$$

$$\leq \mathbb{E}[\,g\, \mathbb{1}_{g \geq a}\,] \to 0 \text{ as } a \to \infty. \quad \therefore \Lambda \text{ is UI.}$$

Eg. If $\Lambda \subseteq L^p$ for some $p > 1$ and $\sup_{X \in \Lambda} \|X\|_{L^p} < \infty$,
then Λ is UI.

$$\mathbb{E}[\,|X| : |X| \geq a\,] \leq \mathbb{E}\left[\,|X| \left(\frac{|X|}{a}\right)^{p-1} : |X| \geq a\,\right] = \frac{1}{a^{p-1}} \mathbb{E}(|X|^p]$$

Eg. Any subset of a UI set is UI.

Lemma: If $\Lambda \subseteq L^1$ is UI, then Λ is L^1-bounded: $\sup_{X\in\Lambda} \mathbb{E}[|X|] < \infty$.

Pf. By assumption, $\underbrace{\sup_{X\in\Lambda} \mathbb{E}[|X| \mathbb{1}_{|X|\geq a}]}_{\leq 1.} \to 0$ as $a \to \infty$.

So fix a s.t.

$$\therefore \ \forall X\in\Lambda, \quad \mathbb{E}[|X|] = \mathbb{E}[|X|(\mathbb{1}_{|X|<a} + \mathbb{1}_{|X|\geq a})]$$
$$= \underbrace{\mathbb{E}[|X|\mathbb{1}_{|X|<a}]}_{\leq a} + \underbrace{\mathbb{E}[|X|\mathbb{1}_{|X|\geq a}]}_{\leq 1} \leq a+1. \quad ///$$

The converse is false, as we'll soon see.

UI is equivalent to another uniform regularity condition.

Def: A collection of random variables $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ is called uniformly absolutely continuous (UAC) if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall B \in \mathcal{F}$, $\mathbb{P}(B) < \delta \Rightarrow \sup_{X\in\Lambda} \mathbb{E}[|X| : B] < \varepsilon$.

i.e. $\limsup_{\delta \downarrow 0} \{\mathbb{E}[|X| : B] \ ; \ X\in\Lambda, \mathbb{P}(B) < \delta\} = 0$.

Prop: For any $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$, Λ is UI iff Λ is UAC and L^1-bounded.

Pf. ($\Longrightarrow$) For any $a>0$, $B \in \mathcal{F}$, $X \in \Lambda$,

$$\mathbb{E}[|X| : B] = \mathbb{E}[|X| : B, |X| \geq a] + \underbrace{\mathbb{E}[|X| : B, |X| < a]}_{\leq a\,\mathbb{P}(B).}$$

$$\therefore \limsup_{\delta \downarrow 0} \{ \mathbb{E}[|X| : B] : X \in \Lambda, \mathbb{P}(B) < \delta \} \leq \limsup_{\delta \downarrow 0} \{ \mathbb{E}[|X| : \cancel{B}, |X| \geq a] : X \in \Lambda, \cancel{\mathbb{P}(B) < \delta} \} + \cancel{\limsup_{\delta \downarrow 0} \{ a\,\mathbb{P}(B) : \mathbb{P}(B) < \delta \}} = 0$$

$\therefore \to 0$ as $a \to \infty$.

We already showed that UI sets are L^1-bounded.

($\Longleftarrow$) Let $K = \sup_{X \in \Lambda} \|X\|_{L^1}$. For $a > 0$, $X \in \Lambda$,

$$\mathbb{P}(|X| \geq a) \leq \frac{\|X\|_{L^1}}{a} \leq \frac{K}{a}.$$

$\forall \varepsilon > 0$, choose $\delta > 0$ s.t. $\mathbb{P}(B) < \delta \Rightarrow \sup_{X \in \Lambda} \mathbb{E}[|X| : B] < \varepsilon$,

Now, set $a = 2K/\delta$, w/ $B = \{|X| \geq a\}$, $\mathbb{P}(B) \leq \frac{K}{a} = \frac{\delta}{2} < \delta$.

$$\Longrightarrow \sup_{X \in \Lambda} \mathbb{E}[|X| : |X| \geq a] < \varepsilon.$$

$$\therefore \lim_{a \to \infty} \left(\qquad \right) = 0.$$

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Cor: If $\Lambda \subseteq L^1$ is UI and $X \in L^1$, then $\Lambda + X = \{Y + X : Y \in \Lambda\}$ is UI.

Pf. By the last proposition, Λ is UAC.

Fix $\varepsilon > 0$, and choose $\delta_1 > 0$ s.t. $\mathbb{P}(B) < \delta_1 \Rightarrow \mathbb{E}[|Y| : B] < \varepsilon/2 \ \forall Y \in \Lambda$.

Of course $\{X\}$ is UI, $\therefore$ UAC, so choose $\delta_2 > 0$ s.t. $\mathbb{P}(B) < \delta_2 \Rightarrow \mathbb{E}[|X| : B] < \frac{\varepsilon}{2}$.

$\therefore$ For $\delta = \delta_1 \wedge \delta_2$, $\forall B \in \mathcal{F}$ w $P(B) < \delta$,

$$\mathbb{E}[|X+Y| : B] \le \mathbb{E}[|X| : B] + \mathbb{E}[|Y| : B] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \therefore \Lambda + X \text{ is UAC.}$$

Also $\sup\{\|X+Y\|_{L^1} : Y \in \Lambda\} \le \sup_{Y \in \Lambda} \|Y\|_{L^1} + \|X\|_{L^1} < \infty$. $\therefore \Lambda + X$ is UI. ///

Uniform Integrability is precisely the gap between L^1-convergence and convergence in probability.

Theorem: (Vitali Convergence Theorem)

Let $\{X_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$, and let X be measurable.

Then

$$X \in L^1 \text{ and } X_n \to X \text{ in } L^1$$

$$\text{iff}$$

$$\{X_n\}_{n=1}^{\infty} \text{ is UI and } X_n \xrightarrow{P} X$$

Pf. If $X_n \to X$ in L^1, then $X_n \to_{\mathbb{P}} X$ [Lec. 13.1]

Let $Y_n = X_n - X \to 0$ in L^1. For any fixed $N \in \mathbb{N}$,

$$\sup_n \mathbb{E}[|Y_n| : |Y_n| \geq a] \leq \underbrace{\sup_{n<N} \mathbb{E}[|Y_n| : |Y_n| \geq a]}_{\text{finite collection, } \therefore \text{ UI}} \vee \sup_{n \geq N} \mathbb{E}[|Y_n| : \cancel{|Y_n| \geq a}]$$

$$\therefore \lim_{a \uparrow \infty} \sup_n \mathbb{E}[|Y_n| : |Y_n| \geq a] \leq \sup_{n \geq N} \mathbb{E}[|Y_n|] \to 0 \text{ as } N \to \infty.$$

Thus $\{Y_n\}_{n=1}^{\infty}$ is UI.

$X_n = Y_n + X$ ← $\{X\}$ is UI.

$\}$ $\therefore \{X_n\}$ is UI

Conversely, suppose $X_n \to_{\mathbb{P}} X$ & $\{X_n\}$ UI.

For $a > 0$, $Y_n \mathbb{1}_{|Y_n| < a} \to 0$ in L^1 by DCT

$$\|X_n - X\|_{L^1} = \mathbb{E}[|Y_n| \mathbb{1}_{|Y_n| < a}] + \mathbb{E}[|Y_n| \mathbb{1}_{|Y_n| \geq a}]$$

$\therefore \limsup_{n \to \infty} (\ \ )$ $\quad \downarrow$ as $n \to \infty$: 0 $\quad \leq \sup_k \mathbb{E}[|Y_k| \mathbb{1}_{|Y_k| \geq a}] \xrightarrow{a \to \infty} 0$.

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Prop: Let $1 \leq p < \infty$ and let $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\Lambda = \{ \mathbb{E}_{\mathcal{G}}[X] : \mathcal{G} \subseteq \mathcal{F} \text{ is a sub-}\sigma\text{-field} \}$$

is L^p-bounded, and UI.

Pf. We've shown $\mathbb{E}_{\mathcal{G}}$ is an L^p-contraction i.e. $\|\mathbb{E}_{\mathcal{G}}[X]\|_{L^p} \leq \|X\|_{L^p}$.
For $p > 1$, it now follows immediately that Λ is UI.

For $p = 1$: we know $|\mathbb{E}_{\mathcal{G}}[X]| \leq \mathbb{E}_{\mathcal{G}}[|X|]$ a.s.

$\therefore$ $\mathbb{E}[\,|\mathbb{E}_{\mathcal{G}}[X]| : |\mathbb{E}_{\mathcal{G}}[X]| \geq a] \leq \mathbb{E}[\mathbb{E}_{\mathcal{G}}[|X|] \mathbb{1}_{|\mathbb{E}_{\mathcal{G}}[X]| \geq a}] = \mathbb{E}[|X| : |\mathbb{E}_{\mathcal{G}}[X]| \geq a]$.

By Markov's inequality, $\mathbb{P}(|\mathbb{E}_{\mathcal{G}}[X]| \geq a) \leq \frac{1}{a} \mathbb{E}[|\mathbb{E}_{\mathcal{G}}[X]|] \leq \frac{1}{a} \mathbb{E}[\mathbb{E}_{\mathcal{G}}[|X|]] = \frac{1}{a} \mathbb{E}[|X|]$. $\to 0$ as $a \to \infty$, unif. in $\mathcal{G}$.

Since $\{X\}$ is UAC, it follows that $\mathbb{E}[|X| : |\mathbb{E}_{\mathcal{G}}[X]| \geq a] \to 0$ as $a \to \infty$ unif. in $\mathcal{G}$.

$$\therefore \lim_{a \to \infty} \sup_{\mathcal{G} \subseteq \mathcal{F}} \mathbb{E}[\,|\mathbb{E}_{\mathcal{G}}[X]| : |\mathbb{E}_{\mathcal{G}}[X]| \geq a] = 0. \quad ///$$

Cor: If $X_n = \mathbb{E}[X | \mathcal{F}_n]$ is a regular martingale, then $\{X_n\}_{n \in \mathbb{N}}$ is UI.

Eg. Let $\{Z_n\}_{n\in\mathbb{N}}$ by iid, $Z_n \overset{d}{=} \frac{1}{2} + \text{Unif}[0,1]$. $\mathbb{E}[Z_n] = \frac{1}{2} + \frac{1}{2} = 1$, $Z_n \geq 0$

we know that $\therefore$ $X_n = Z_0 Z_1 \cdots Z_n$ is a martingale, and $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] = 1$ $\forall n$.

Is it regular? $X_n \overset{?}{=} \mathbb{E}[X \mid \mathcal{F}_n]$ for some $X \in L^1$ $\therefore$ $X \geq 0$.

Notice: $\frac{1}{n} \ln X_n = \frac{1}{n} \sum_{j=0}^{n} \ln Z_j$ $\qquad$ $\ln Z_j = \ln\left(\frac{1}{2} + U_j\right) \in \left(\ln\frac{1}{2}, \ln\frac{3}{2}\right]$ bded $\therefore L^1$.

$\rightarrow -0.045$ a.s. < 0. $\qquad$ $\mathbb{E}[\ln Z_j] = \int_0^1 \ln\left(\frac{1}{2}+u\right) du \doteq -0.045$

$\therefore$ $X_n \rightarrow 0$ a.s. $\therefore$ $X_n \rightarrow_p 0$.

If X_n were regular, $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for some $X \in L^1$, then $\{X_n\}$ would be UI.

$\therefore$ By Vitali, $X_n \rightarrow 0$ in L^1. But then

$\therefore$ $\|X\|_{L^1} = \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_n]] = \mathbb{E}[X_n] \rightarrow 0$ as $n \rightarrow \infty$.

i.e. $X = 0$. $\therefore$ $X_n = \mathbb{E}_{\mathcal{F}_n}[0] = 0$ $\forall n$.

We see that L^1-boundedness $\not\Rightarrow$ UI, and there are ≥ 0, L^1-bounded martingales that are not regular.