Martingales

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ filtered probability space. S a Banach space; usually $\mathbb{R}^d$.
(Equipped w/ $\mathcal{B}(S)$.)

Def: An adapted process $(X_t)_{t\in T}$ in $L^1(\Omega, (\mathcal{F}_t)_{t\in T}, \mathbb{P}; S)$ is a
martingale if $\mathbb{E}[X_t | \mathcal{F}_s] = X_s \quad \forall s<t$
" " if " " " "
" " if " " " "

Thinking in terms of earnings while betting on a gambling game,

In the special case $T = \mathbb{N}$ (which we'll focus on for now), by induction the (sub/super) martingale property reduces to

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$$

We also have the equivalent forms

$\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0$, or $\mathbb{E}[X_t | \mathcal{F}_s] =$

If $(X_t)_{t\in T}$ is a martingale, $\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t \mid \mathcal{F}_s]] = \mathbb{E}[X_s]$ $\forall s \leq t$

We can shrink the filtration down to (anything containing) $(\mathcal{F}^X_n)_{n\in\mathbb{N}}$.

Lemma: Let $(X_t)_{t\in T}$ be a martingale wrt $(\mathcal{F}_t)_{t\in T}$.

Let $(\mathcal{G}_t)_{t\in T}$ be a filtration with $\mathcal{F}_t \supseteq \mathcal{G}_t \supseteq \mathcal{F}^X_t$

Then $(X_t)_{t\in T}$ is a martingale wrt $(\mathcal{G}_t)_{t\in T}$.

Pf. By definition X_t is $\mathcal{F}^X_t$-measurable.

$\therefore$ $(X_t)_{t\in T}$ is $(\mathcal{G}_t)_{t\in T}$-adapted.

For $s \leq t$, $\mathbb{E}[X_t \mid \mathcal{G}_s]$

Examples.

1. Let $(X_t)_{t\in T}$ be an adapted process with independent increments.
If $\mathbb{E}[X_t - X_s] = 0$ $\forall s \le t$, then $(X_t)_{t\in T}$ is a martingale.

b/c $\mathbb{E}[X_t | \mathcal{F}_s] =$

- (pre-) Brownian motion.

  $X_t - X_s \overset{d}{=} \mathcal{N}(0, t-s)$, mean 0

  $\therefore$ martingale.

- Poisson process $N_t - N_s \overset{d}{=} \text{Poiss}(\lambda(t-s))$ $\therefore$ $\mathbb{E}(N_t - N_s) \ge 0$ submartingale

  ↳ Compensated Poisson process

  $X_t = N_t - \lambda t$

  martingale.

- $\{\xi_k\}_{k\in\mathbb{N}}$ independent L^1 rv's, $X_n = \sum_{k=0}^{n} \xi_k$.

  ↳ if $\mathbb{E}[\xi_k] = 0$ $\forall k$, martingale. (Eg SRW)

  ↳ if $\mathbb{E}[\xi_k] \ge 0$ $\forall k$, submartingale. (RW $p > \frac{1}{2}$)

  ↳ if $\mathbb{E}[\xi_k] \le 0$ $\forall k$, supermartingale. (RW $p < \frac{1}{2}$)

2. A gambler's earnings employing a betting strategy in a casino. [Lec 47.1]

3. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. For the given filtration $\{\mathcal{F}_t\}_{t\in T}$, define

$$X_t := \mathbb{E}[X \mid \mathcal{F}_t]$$

We know $\mathbb{E}_{\mathcal{F}_t}$ is an L^1 contraction, so $\|X_t\|_{L^1} \leq \|X\|_{L^1} < \infty$.

By definition, $\mathbb{E}[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-measurable; $\therefore (X_t)_{t\in T}$ is adapted.

For $s \leq t$, $\mathbb{E}[X_t \mid \mathcal{F}_s]$

A martingale of this form is called a regular martingale.

↳ Not all martingales are regular.

Eg. SRW. If $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for $X \in L^1$, then

$\infty > \sup_n \mathbb{E}[|X_n|]$, $\therefore$ if $b_n \uparrow \infty$, $\mathbb{E}[|X_n / b_n|] \to 0$.

4. Product martingales. $\{Z_n\}_{n=0}^{\infty}$ independent, L^1. Set $X_n = Z_1 \cdots Z_n$

$\mathcal{F}_n := \sigma(Z_0, Z_1, \dots, Z_n)$

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n]$$

Thus $(X_n)_{n\geq 0}$ is a martingale iff

If we take $Z_n \geq 0$ a.s. then $(X_n)_{n\geq 0}$ is also ≥ 0 a.s., and

is a sub/super martingale

iff

In the case $(X_n)_{n\geq 0}$ is a martingale, $\mathbb{E}[|X_n|] = 1 \ \forall n$,

so the process is L^1-bounded (unlike SRW).

But does that mean it is regular?

I.e. ¿ $\exists X \in L^1$ s.t. $X_n = \mathbb{E}[X | \mathcal{F}_n]$?