Martingales

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ filtered probability space. S a Banach space; usually $\mathbb{R}^d$.
(Equipped w/ $\mathcal{B}(S)$.)

Def: An adapted process $(X_t)_{t\in T}$ in $L^1(\Omega, (\mathcal{F}_t)_{t\in T}, \mathbb{P}; S)$ is a

martingale if $\mathbb{E}[X_t | \mathcal{F}_s] = X_s \quad \forall s<t$ — fair games

sub " " if " " $\geq$ " " — favors the player

super " " if " " $\leq$ " " — favors the house.

Thinking in terms of earnings while betting on a gambling game,

In the special case $T = \mathbb{N}$ (which we'll focus on for now), by induction the (sub/super) martingale property reduces to

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] \overset{(\geq/\leq)}{=} X_n$$

We also have the equivalent forms

$\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \overset{(\geq/\leq)}{=} 0$, or $\mathbb{E}[X_t | \mathcal{F}_s] \overset{(\geq/\leq)}{=} X_{s\wedge t}$

If $(X_t)_{t\in T}$ is a sub / super martingale, $\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t | \mathcal{F}_s]] = \mathbb{E}[X_s]$ $\forall s \le t$

$\geq$ / $\leq$

$\mathbb{E}[X_t] \uparrow$

$\mathbb{E}[X_t] \downarrow$

We can shrink the filtration down to (anything containing) $(\mathcal{F}^X_n)_{n\in\mathbb{N}}$.

Lemma: Let $(X_t)_{t\in T}$ be a (sub/super) martingale wrt $(\mathcal{F}_t)_{t\in T}$.

Let $(\mathcal{G}_t)_{t\in T}$ be a filtration with $\mathcal{F}_t \supseteq \mathcal{G}_t \supseteq \mathcal{F}^X_t = \sigma(X_s : s \le t)$

Then $(X_t)_{t\in T}$ is a (sub/super) martingale wrt $(\mathcal{G}_t)_{t\in T}$.

Pf. By definition X_t is $\mathcal{F}^X_t$-measurable. $\mathcal{G}_t \supseteq \mathcal{F}^X_t$

$\therefore (X_t)_{t\in T}$ is $(\mathcal{G}_t)_{t\in T}$-adapted.

For $s \le t$, $\mathbb{E}[X_t | \mathcal{G}_s] = \mathbb{E}_{\mathcal{G}_s}[\mathbb{E}_{\mathcal{F}_s}[X_t]]$ b/c $\mathcal{G}_s \subseteq \mathcal{F}_s$

$= \mathbb{E}_{\mathcal{G}_s}[X_s] = X_s$.

$(\geq / \leq)$

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Examples.

1. Let $(X_t)_{t\in T}$ be an adapted process with independent increments.

If $\mathbb{E}[X_t - X_s] = 0$ $(\geq 0 / \leq 0)$ $\forall s \leq t$, then $(X_t)_{t\in T}$ is a (sub/super) martingale.

b/c $\mathbb{E}[X_t | \mathcal{F}_s] = \mathbb{E}[X_t - X_s + X_s | \mathcal{F}_s] = \underbrace{\mathbb{E}[X_t - X_s | \mathcal{F}_s]}_{\mathbb{E}[X_t - X_s] = 0 \, (\geq / \leq)} + \underbrace{\mathbb{E}[X_s | \mathcal{F}_s]}_{X_s}$

- (pre-) Brownian motion.

  $X_t - X_s \overset{d}{=} \mathcal{N}(0, t-s)$, mean 0

  $\therefore$ martingale.

- Poisson process $N_t - N_s \overset{d}{=} \text{Poiss}(\lambda(t-s))$ $\therefore$ $\mathbb{E}(N_t - N_s) \geq 0$ submartingale

  ↳ Compensated Poisson process

  $X_t = N_t - \lambda t$ $\quad X_t - X_s = N_t - N_s - \lambda(t-s)$ indep.

  martingale. $\mathbb{E}[X_t - X_s] = 0$.

- $\{\xi_k\}_{k\in\mathbb{N}}$ independent L^1 rv's, $X_n = \sum_{k=0}^{n} \xi_k$.

  ↳ if $\mathbb{E}[\xi_k] = 0$ $\forall k$, martingale. (Eg SRW)

  ↳ if $\mathbb{E}[\xi_k] \geq 0$ $\forall k$, submartingale. (RW $p > \frac{1}{2}$)

  ↳ if $\mathbb{E}[\xi_k] \leq 0$ $\forall k$, supermartingale. (RW $p < \frac{1}{2}$)

2. A gambler's earnings employing a betting strategy in a casino. [Lec 47.1]

3. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. For the given filtration $\{\mathcal{F}_t\}_{t \in T}$, define

$$X_t := \mathbb{E}[X \mid \mathcal{F}_t]$$ (came up in our analysis of $\mathbb{E}[\cdot \mid \mathcal{F}_\tau]$.

We know $\mathbb{E}_{\mathcal{F}_t}$ is an L^1 (L^p) contraction, so $\|X_t\|_{L^1} \leq \|X\|_{L^1} < \infty$.

By definition, $\mathbb{E}[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-measurable; $\therefore (X_t)_{t \in T}$ is adapted.

For $s \leq t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}_{\mathcal{F}_s}[\mathbb{E}_{\mathcal{F}_t}[X]] = \mathbb{E}_{\mathcal{F}_s}[X] = X_s$. $\therefore$ martingale.

A martingale of this form is called a regular martingale.

↳ Not all martingales are regular.

Eg. SRW. If $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for $X \in L^1$, then

$\infty > \sup_n \mathbb{E}[|X_n|]$, $\therefore$ if $b_n \uparrow \infty$, $\mathbb{E}[|X_n / b_n|] \to 0$.

$X_n = \sum_{k=0}^{n} \xi_k$ iid ± 1 RV's

$\therefore$ by CLT, $\frac{X_n}{\sqrt{n}} \to_w N(0,1)$.

4. Product martingales. $\{Z_n\}_{n=0}^{\infty}$ independent, L^1. Set $X_n = Z_1 \cdots Z_n$

$\mathcal{F}_n := \sigma(Z_0, Z_1, \ldots, Z_n) \supseteq \sigma(X_0, X_1, \ldots, X_n)$

$\mathbb{E}[|X_n|] = \mathbb{E}[|Z_1| \cdots |Z_n|]$

$= \mathbb{E}[|Z_1|] \cdots \mathbb{E}[|Z_n|] < \infty$.

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_n Z_{n+1} | \mathcal{F}_n] = X_n \mathbb{E}[Z_{n+1} | \mathcal{F}_n]$$

$$= X_n \mathbb{E}[Z_{n+1}].$$

$= 1$.

Thus $(X_n)_{n\geq 0}$ is a martingale iff $\mathbb{E}[Z_n] = 1 \quad \forall n \geq 1$

If we take $Z_n \geq 0$ a.s. then $(X_n)_{n\geq 0}$ is also ≥ 0 a.s., and is a sub/super martingale iff $\mathbb{E}[Z_n] \geq / \leq 1 \quad \forall n$.

In the case $(X_n)_{n\geq 0}$ is a martingale, $\mathbb{E}[|X_n|] = 1 \ \forall n$, so the process is L^1-bounded (unlike SRW).

But does that mean it is regular?

I.e. ¿ $\exists X \in L^1$ s.t. $X_n = \mathbb{E}[X | \mathcal{F}_n]$?

No.