Let's play a simple gambling game (like roulette) again...
but this time with a different betting strategy.

Martingale Betting Strategy

- when you win, bet $\$1$ next.
- when you lose, double your bet from last round.

Eg. Outcome: W W L W L L W ...

Bet: 1 ↓ 1

Total: 2

The idea: when you have a sequence of k losses, followed by a win, your total winnings over these $k+1$ rounds are

$$-1-2-4-\cdots-2^{k-1}+2^k$$

Thus, if F_0 is your initial fortune, and τ_n is the time of the n^{th} win, $F_{\tau_n} =$ But...

Employing the Martingale betting strategy, your maximum possible fortune after n rounds is $F_0 + n$. Meanwhile, your maximum possible losses are staggering. In a k-round losing streak, you lose $2^k - 1$. This can easily wipe out your entire fortune, even for relatively small k. And (by LLN) there will eventually be a losing streak of any given size.

Moral: If you have infinite initial fortune (or loan shark...) the Martingale betting strategy is guaranteed to work. Otherwise, it's really dumb.

Historical Note: Why "Martingale"?

It likely comes from an 18th Century Provencal expression "jouga a la martegalo", meaning "play like an idiot".

Likely derives from Martigues, a township near Marseille.

Betting Strategies

The game: at each turn, some state s is chosen from a finite state space S.

Eg. tossing a coin, $S = \{H, T\}$

tossing two dice, $S = \{2, 3, \ldots, 12\}$

S comes with a probability distribution ν.

The game is a sequence $\{Z_n\}_{n=1}^{\infty}$ of iid S-valued rv's w $Z_n \overset{d}{=} \nu$.

The player: The player forms a "strategy" which may have some random input W "whims" (a rv on the same prob. space as the game, but some other abstract state space). $W, \{Z_n\}_{n=1}^{\infty}$ independent.

S/he places bets B_n.

B_n

$= B_n($ $) \Leftarrow$

- ↳ B_n is the bet on the n^{th} round, placed <u>before</u> the n^{th} play
- ↳ B_n may depend on the current state of the game, the whims, and the previous outcomes.

The game: $\{Z_n\}_{n=1}^{\infty}$ i.i.d w/ distribution ν on S.

The player: whims W indep. $\{Z_n\}_{n=1}^{\infty}$. Bets $B_n = B_n(W, Z_1, \dots, Z_{n-1}, S)$

The house: Sets the odds ν, and also the payout function

$\alpha : S \to [0, \infty)$ – for each \$1 the player bets on s,

the house pays out $\$\alpha(s)$ when s is "rolled".

Let X_n = the player's fortune right after the n^{th} round.

The player's winnings/losses in the n^{th} round:

$X_k - X_{k-1} = -$ (total bet on k^{th} round) + (total payout on k^{th} round)

$\therefore X_n = X_0 + \sum_{k=1}^{n} ($ " $\qquad$ " $)$

$\uparrow$ initial fortune

We'd like to analyze $(X_n)_{n \in \mathbb{N}}$

Basic question: do you <u>expect</u> to increase your fortune at each round?

I.e. $\mathbb{E}[X_n - X_{n-1}] \geq 0$?

Let $\mathcal{F}_k = \sigma(W, Z_1, \dots, Z_k)$

$$X_n - X_{n-1} = -\sum_{s \in S} B_n(W, Z_1, \dots, Z_{n-1}, s) + B_n(W, Z_1, \dots, Z_{n-1}, Z_n)\,\alpha(Z_n)$$

$$\mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] =$$

$$\mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] = -\sum_{s\in S} B_n(W, Z_1, \dots, Z_{n-1}, s) + \sum_{s \in S} B_n(W, Z_1, \dots, Z_{n-1}, s)\, \alpha(s)\nu(s)$$

$$= \sum_{s\in S} B_n(W, Z_1, \dots, Z_{n-1}, s)\big(\alpha(s)\nu(s) - 1\big)$$

Prop: If the house sets the game odds ν and payout function α s.t.

$$\alpha(s)\nu(s) \left\{ \qquad \right\}, \quad \text{then} \quad \mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] \left\{ \qquad \right\}$$

E.g. The Martingale Betting Strategy.

$S = \{H, T\}$ $\quad \nu = \mathrm{Unif}(S)$ $\quad \alpha(s) =$

$\therefore\ \alpha(s)\nu(s) =$