Let's play a simple gambling game (like roulette) again...
but this time with a different betting strategy.

Martingale Betting Strategy

- when you win, bet $\$1$ next.
- when you lose, double your bet from last round.

Eg.

Outcome:	W	W	L	W	L	L	W	...
Bet:	1	1	1	2	1	2	4	
Total:	2	3	2	4	3	1	5	

The idea: when you have a sequence of k losses, followed by a win, your total winnings over these $k+1$ rounds are

$$-1-2-4-\cdots-2^{k-1}+2^{k}=1.$$

Thus, if F_0 is your initial fortune, and τ_n is the time of the n^{th} win, $F_{\tau_n}=F_0+n$. But...

Employing the Martingale betting strategy, your maximum possible fortune after n rounds is $F_0 + n$. Meanwhile, your maximum possible losses are staggering. In a k-round losing streak, you lose $2^k - 1$. This can easily wipe out your entire fortune, even for relatively small k. And (by LLN) there will eventually be a losing streak of any given size.

Moral: If you have infinite initial fortune (or loan shark...) the Martingale betting strategy is guaranteed to work. Otherwise, it's really dumb.

Historical Note: Why "Martingale"?

It likely comes from an 18th Century Provencal expression "jouga a la martegalo", meaning "play like an idiot".

Likely derives from Martigues, a township near Marseille.

"Village of Idiots"

Betting Strategies

The game: at each turn, some state s is chosen from a finite state space S.

Eg. tossing a coin, $S = \{H, T\}$

tossing two dice, $S = \{2, 3, \ldots, 12\}$ $\quad \nu(2) = \nu(12) = \frac{1}{36}$, $\nu(3) = \nu(11) = \frac{1}{18}$ ...

S comes with a probability distribution ν.

The game is a sequence $\{Z_n\}_{n=1}^\infty$ of iid S-valued rv's w $Z_n \stackrel{d}{=} \nu$.

The player: The player forms a "strategy" which may have some random input W "whims" (a rv on the same prob. space as the game, but some other abstract state space). $W, \{Z_n\}_{n=1}^\infty$ independent.

S/he places bets B_n.

$B_n = B_n(W, Z_1, \ldots, Z_{n-1}, s)$ $\Leftarrow$

- B_n is the bet on the n^{th} round, placed <u>before</u> the n^{th} play
- B_n may depend on the current state of the game, the whims, and the previous outcomes.

meas. wrt. $\sigma(W, Z_1, \ldots, Z_{n-1})$

The game: $\{Z_n\}_{n=1}^{\infty}$ i.i.d w distribution ν on S.

The player: whims W indep. $\{Z_n\}_{n=1}^{\infty}$. Bets $B_n = B_n(W, Z_1, \dots, Z_{n-1}, S)$

The house: Sets the odds ν, and also the payout function

$\alpha : S \to [0, \infty)$ – for each \$1 the player bets on s, the house pays out $\$\alpha(s)$ when s is "rolled".

Let X_n = the player's fortune right after the n^{th} round.

The player's winnings/losses in the n^{th} round:

$$X_k - X_{k-1} = -(\text{total bet on } k^{th} \text{ round}) + (\text{total payout on } k^{th} \text{ round})$$

$$= -\sum_{s \in S} B_k(W, Z_1, \dots, Z_{n-1}, s) + B_k(W, Z_1, \dots, Z_{k-1}, Z_k)\,\alpha(Z_k)$$

$$\therefore X_n = \underbrace{X_0}_{\text{initial fortune}} + \sum_{k=1}^{n} \left(\text{''} \quad\quad \text{''} \right)$$

We'd like to analyze $(X_n)_{n \in \mathbb{N}}$

Basic question: do you <u>expect</u> to increase your fortune at each round?

I.e. $\mathbb{E}[X_n - X_{n-1}] \geq 0$?

$$\overset{\shortparallel}{\mathbb{E}[\mathbb{E}[X_n - X_{n-1} | \mathcal{G}]]} \leftarrow \text{chose } \mathcal{G} \text{ to help.}$$

Let $\mathcal{F}_k = \sigma(W, Z_1, \dots, Z_k)$

$$X_n - X_{n-1} = -\sum_{s \in S} \underbrace{B_n(W, Z_1, \dots, Z_{n-1}, s)}_{\mathcal{F}_{n-1}\text{-meas.}} + \underbrace{B_n(W, Z_1, \dots, Z_{n-1}, Z_n)\,\alpha(Z_n)}_{\mathcal{F}_n\text{-meas.}}$$

$$\mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] = -\sum_{s \in S} B_n(W, Z_1, \dots, Z_{n-1}, s)$$

$$+ \mathbb{E}_{\mathcal{F}_{n-1}}[B_n(\underbrace{W, Z_1, \dots, Z_{n-1}}_{\mathcal{F}_{n-1}}, \underbrace{Z_n)\,\alpha(Z_n)}_{\text{indep. from } \mathcal{F}_{n-1}}]$$

$$= \mathbb{E}_{Law(Z_n)}[B_n(\text{———}, \cdot)\,\alpha(\cdot)]\Big|_{(\text{—})=(W, Z_1, \dots, Z_{n-1})}$$

$$= \sum_{s \in S} B_n(W, Z_1, \dots, Z_{n-1}, s)\,\alpha(s)\,\nu(s).$$

$$\mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] = -\sum_{s\in S} B_n(w, z_1, \dots, z_{n-1}, s) + \sum_{s \in S} B_n(w, z_1, \dots, z_{n-1}, s)\, \alpha(s) \nu(s)$$

$$= \sum_{s\in S} \underbrace{B_n(w, z_1, \dots, z_{n-1}, s)}_{\geq 0} \left(\alpha(s)\nu(s) - 1\right)$$

boils down to the sign of

Prop: If the house sets the game odds ν and payout function α s.t.

$$\alpha(s)\nu(s) \begin{Bmatrix} \geq 1 \\ = 1 \\ \leq 1 \end{Bmatrix}, \quad \text{then} \quad \mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] \begin{Bmatrix} \geq 0 \\ = 0 \\ \leq 0 \end{Bmatrix}$$

Eg. The Martingale Betting Strategy.

$S = \{H, T\}$ $\quad \nu = \mathrm{Unif}(S)$ $\quad \alpha(s) = 2$

$\therefore \alpha(s)\nu(s) = 2 \cdot \frac{1}{2} = 1 \quad \forall s$

$\therefore \mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] = 0 \quad \forall n$

$\therefore \mathbb{E}[X_n] = \mathbb{E}[X_{n-1}]$ "fair game".