How many times do you need to toss a fair coin to get 10 Heads?

If we toss it n times, $\mathbb{E}(\#\text{Heads}) = $

So, we want $\frac{n}{2} = 10$, $n = 20$ ... But that's not really answering the question.

$\{X_1, X_2, \dots\}$ i.i.d. $\text{Ber}(\frac{1}{2})$

$$\tau_k = \inf\{n \geq 0 : \sum_{j=1}^{n} X_j = k\}$$

The question is: what is $\mathbb{E}[\tau_{10}]$?

Theorem: (Wald's Identity)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables, and let τ be a stopping time relative to $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. If $f \geq 0$, or if $f(X_n) \in L^1$ and $\mathbb{E}[\tau] < \infty$, then

$$\mathbb{E}\left[\sum_{n=1}^{\tau} f(X_n)\right] =$$

Pf. First, assume $f \geq 0$.

$$\mathbb{E}\left[\sum_{n=1}^{\tau} f(X_n)\right] = \mathbb{E}\left[\sum_{n=1}^{\infty} f(X_n) \mathbb{1}_{n \leq \tau}\right]$$

$$= \mathbb{E}[f(X_1)] \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}_{n \leq \tau}]$$

Eg. Roll a die, yielding some value $D \in \{1,2,3,4,5,6\}$.
Now roll the die D times, and add up the D values.
What's the expected sum of these D rolls?

E.g. (Gambler's Ruin, revisited)

Let $(X_n)_{n=1}^{\infty}$ be a random walk on $\mathbb{Z}$, $\mathbb{P}(X_{n+1}=k+1 \mid X_n=k) = p \in (0,1)$.

Then we can construct it as $X_n = \sum_{k=1}^{n} \xi_k$

How long does it take, starting at 0, to reach $k \neq 0$?

$\tau = \inf\{n \geq 1 : X_n = k\}$

$\therefore \quad \mathbb{E}[\tau] = \dfrac{k}{p-(1-p)}$! ... But this is <0 if k, $p-\frac{1}{2}$ have opposite signs.

With $p=\frac{1}{2}$, we can conclude $\mathbb{E}[\tau] = \infty$.