How many times do you need to toss a fair coin to get 10 Heads?

If we toss it n times, $\mathbb{E}(\#\text{Heads}) = \mathbb{E}\left[\sum_{j=1}^{n} X_j\right] = \sum_{j=1}^{n} \mathbb{E}[X_j] = \frac{n}{2}$.

$X_j \overset{d}{=} \mathrm{Ber}\left(\frac{1}{2}\right)$

So, we want $\frac{n}{2} = 10$, $n = 20$ ... But that's not really answering the question.

$\{X_1, X_2, \dots\}$ i.i.d. $\mathrm{Ber}\left(\frac{1}{2}\right)$ $\quad \mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$

$$\tau_k = \inf\left\{ n \geq 0 : \sum_{j=1}^{n} X_j = k \right\}$$

The question is: what is $\mathbb{E}[\tau_{10}]$?

$$\{\tau_k = n\} = \left\{ \sum_{j=1}^{n} X_j = k, \sum_{j=1}^{n-1} X_j = k-1 \right\} \in \mathcal{F}_n.$$

stopping time.

Theorem: (Wald's Identity)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables, and let τ be a stopping time relative to $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. If $f \geq 0$, or if $f(X_n) \in L^1$ and $\mathbb{E}[\tau] < \infty$, then

$$\mathbb{E}\left[\sum_{n=1}^{\tau} f(X_n)\right] = \mathbb{E}[f(X_1)]\,\mathbb{E}[\tau].$$

Pf. First, assume $f \geq 0$.

$$\mathbb{E}\left[\sum_{n=1}^{\tau} f(X_n)\right] = \mathbb{E}\left[\sum_{n=1}^{\infty} f(X_n)\,\mathbb{1}_{n\leq\tau}\right] \overset{\text{Fubini/Tonelli}}{=} \sum_{n=1}^{\infty} \mathbb{E}[f(X_n)\mathbb{1}_{n\leq\tau}].$$

$\{n \leq \tau\} = \{\tau \leq n-1\}^c \in \mathcal{F}_{n-1}$

$\therefore$ by Doob-Dynkin, $\mathbb{1}_{n\leq\tau} = F_n(X_1, \dots, X_{n-1})$

$$= \sum_{n=1}^{\infty} \mathbb{E}[f(X_n) F_n(X_1, \dots, X_{n-1})].$$

$$= \sum_{n=1}^{\infty} \mathbb{E}[f(X_n)]\,\mathbb{E}[F_n(X_1, \dots, X_{n-1})]$$

$$= \mathbb{E}[f(X_1)] \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}_{n\leq\tau}]$$

$$\sum_{n=1}^{\infty} \sum_{k\geq n} \cancel{\mathbb{E}[\mathbb{1}_{\tau=k}]}\ \mathbb{P}(\tau=k)$$

$$= \sum_{k=1}^{\infty} \sum_{n\leq k} \mathbb{P}(\tau=k)$$

$$= \sum_{k=1}^{\infty} k\,\mathbb{P}(\tau=k)$$

$$= \mathbb{E}[\tau].$$

Repeat, apply to $|f|$.

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Eg. Roll a die, yielding some value $D \in \{1,2,3,4,5,6\}$.
Now roll the die D times, and add up the D values.
What's the expected sum of these D rolls?

$D_0 = D$, $\{D_n\}_{n=0}^{\infty}$ iid die rolls $\mathcal{F}_n = \sigma(D_0, D_1, \dots, D_n)$.

$D_n \overset{d}{=} \mathrm{Unf}\{1, \dots, 6\}$.

$\{D = k\} \in \mathcal{F}_0 \subseteq \mathcal{F}_k$ $\therefore$ D is a stopping time.

$$E\left[\sum_{j=0}^{D} D_j\right] = E[D_0]\,E[D] = E[D]^2$$

$$\parallel$$

$$E[D_0] + E\left[\sum_{j=1}^{D} D_j\right]$$

$$\hookrightarrow = E[D]^2 - E[D]$$

$$= 8.75.$$

Eg. (Gambler's Ruin, revisited)

Let $(X_n)_{n=1}^{\infty}$ be a random walk on $\mathbb{Z}$, $\mathbb{P}(X_{n+1}=k+1 \mid X_n = k) = p \in (0,1)$.

Then we can construct it as $X_n = \sum_{k=1}^{n} \xi_k$ $\qquad \xi_k$ i.i.d, $\xi_k \overset{d}{=} p\delta_1 + (1-p)\delta_{-1}$

How long does it take, starting at 0, to reach $k \neq 0$? $\qquad X_0 = 0$.

$\tau = \inf\{n \geq 1 : X_n = k\}$

$$k = \mathbb{E}[X_\tau] = \mathbb{E}\Big[\sum_{n=1}^{\tau} \xi_n\Big] = \mathbb{E}[\xi_1]\,\mathbb{E}[\tau] = (p-(1-p))\,\mathbb{E}[\tau].$$

$\therefore \quad \mathbb{E}[\tau] = \dfrac{k}{p-(1-p)}$! ... But this is <0 if k, $p-\frac{1}{2}$ have opposite signs.

If $\mathbb{E}[\tau] < \infty$, then $k = \underbrace{\mathbb{E}[\xi_1]}_{p=\frac{1}{2}:\ 0}\,\mathbb{E}[\tau]$

with $p=\frac{1}{2}$, we can conclude $\mathbb{E}[\tau] = \infty$.