We've seen [Lec 44.2] that any irreducible, finite state Markov chain $(X_n)_{n\geq 0}$ is positive recurrent:

and $(X_n)_{n\geq 0}$ has a unique invariant distribution

$$\mu_i := \frac{1}{\mathbb{E}^i[\tau_i]}.$$

These weights have a precise meaning.

Theorem: (Ergodic theorem) Let $V_j(N)$ be the number of times $(X_n)_{n\geq 0}$ visits j in the first N steps.

$$V_j(N) = \sum_{n=0}^{N} \mathbb{1}_{X_n = j}.$$

Then the proportion of time spent in state j converges to μ_j, $\mathbb{P}^i$ - a.s.

$$\mathbb{P}^i\left(\lim_{N\to\infty} \frac{V_j(N)}{N} = \frac{1}{\mathbb{E}^i[\tau_i]}\right) = 1, \quad \forall i,j.$$

$\frac{V_j(N)}{N} \to \mu_j \quad P^i\text{-a.s.}, \quad \therefore \quad \mathbb{E}^i\left[\frac{V_j(N)}{N}\right] \to \mu_j$

It's not hard to see that if $q^n(i,j) \xrightarrow{n\to\infty} \nu_j \ \forall i$, then ν is invariant.
The Ergodic theorem is a kind of converse; but it does not imply
Indeed, that kind of pointwise convergence is just not true in general.

Eg. $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Unique invariant distribution $\mu =$

and the chain is positive recurrent with

$\mathbb{E}^i[T_i] =$

But, for all i, $(q^n(i,j))_{n=0}^{\infty}$ alternates between $0, 1$. The chain is periodic.

To prove the Ergodic theorem, we use the strong Markov property.

Def. Let $\sigma_j^{(n)}$ be the n^{th} excursion time to state j:

$$\sigma_j^{(1)} := \inf\{n \geq 1 : X_n = j\}$$
$$\sigma_j^{(n)} := \inf\{n \geq 1 :$$

} The time it takes to return to j after the previous visit.

Lemma: Relative to $\mathbb{P}^i$ for any i, $\{\sigma_j^{(n)}\}_{n=1}^{\infty}$ are independent, and $\{\sigma_j^{(n)}\}_{n=2}^{\infty}$ are iid with

$$\text{Law}_{\mathbb{P}^i}(\sigma_j^{(n)}) = \text{Law}_{\mathbb{P}^j}(\tau_j) \quad \forall i, \ \forall n \geq 2.$$

Pf. Let $\tau_j^{(n)} = \inf\{n > \tau_j^{(n-1)} : X_n = j\}$, where $\tau_j^{(0)} = 0$.

I.e. $\tau_j^{(n)}$ is the n^{th} passage time to j. stopping time.

Note that

$$\sigma_j^{(n+1)}(X_0, X_1, X_2, \ldots) =$$

$$\sigma_j^{(n+1)}(X_0, X_1, X_2, \ldots) = \sigma_j^{(1)}(X_{\tau_j^{(n)}}, X_{\tau_j^{(n)}+1}, X_{\tau_j^{(n)}+2}, \ldots)$$

Since $X_{\tau_j^{(n)}} = j$, and since $\tau_j^{(n)}$ is a stopping time, by the Strong Markov property:

- $\sigma_j^{(n+1)}$ is independent from $(X_0, X_1, \ldots, X_{\tau_j^{(n)}})$
  - ↳ But $\sigma_j^{(1)}, \ldots, \sigma_j^{(n)}$ are functions of
- As $X_{\tau_j^{(n)}} = j$, $(X_{\tau_j^{(n)}}, X_{\tau_j^{(n)}+1}, \ldots) \stackrel{d}{=} (X_0, X_1, \ldots)$ under $\mathbb{P}^j$.

Pf. of Ergodic theorem: By SLLN:

$$\forall i, \quad \lim_{N \to \infty} \frac{1}{N} \left[\sigma_j^{(1)} + \sigma_j^{(2)} + \cdots + \sigma_j^{(N)} \right]$$

$$= \mathbb{E}^i[\sigma_j^{(2)}] = \mathbb{E}^j[\tau_j] \qquad \mathbb{P}^i\text{-a.s.}$$

But $\sigma_j^{(1)} + \sigma_j^{(2)} + \cdots + \sigma_j^{(N)}$

$V_j(N) = \sum_{n=0}^{\infty} \mathbb{1}_{X_n = j}$ = # visits to j during first N steps.

$$\therefore \quad \tau_j^{V_j(N)} \leq N < \tau_j^{V_j(N)+1}$$