We've seen [Lec 44.2] that any irreducible, finite state Markov chain $(X_n)_{n\geq 0}$ is positive recurrent: $\mathbb{E}^i[\tau_i] < \infty \quad \forall i$

and $(X_n)_{n\geq 0}$ has a unique invariant distribution

$$\mu_i := \frac{1}{\mathbb{E}^i[\tau_i]} .$$

These weights have a precise meaning.

Theorem: (Ergodic theorem) Let $V_j(N)$ be the number of times $(X_n)_{n\geq 0}$ visits j in the first N steps.

$$V_j(N) = \sum_{n=0}^{N} \mathbb{1}_{X_n = j} .$$

Then the proportion of time spent in state j converges to μ_j, $\mathbb{P}^i$ - a.s.

$$\mathbb{P}^i\left(\lim_{N\to\infty} \frac{V_j(N)}{N} = \frac{1}{\mathbb{E}^i[\tau_j]} \right) = 1, \quad \forall i, j .$$

$\frac{V_j(N)}{N} \to \mu_j \quad P^i\text{-a.s.}, \quad \therefore \quad \mathbb{E}^i\left[\frac{V_j(N)}{N}\right] \to \mu_j$

$$\| $$

$$\frac{1}{N}\mathbb{E}^i\left[\sum_{n=0}^{N} \mathbb{1}_{X_n=j}\right] = \frac{1}{N}\sum_{n=0}^{N} \underbrace{P^i(X_n=j)}_{q^n(i,j)}$$

It's not hard to see that if $q^n(i,j) \xrightarrow{n\to\infty} \nu_j \ \forall i$, then ν is invariant. The Ergodic theorem is a kind of converse; but it does not imply. Indeed, that kind of pointwise convergence is just not true in general.

Eg. $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \therefore \quad q^n = \begin{cases} I & n \text{ even} \\ q & n \text{ odd}. \end{cases}$

Unique invariant distribution $\mu = [\tfrac{1}{2} \ \tfrac{1}{2}]$.

and the chain is positive recurrent with

$$\mathbb{E}^i[T_i] = 2 \qquad i \in \{1, 2\}.$$

But, for all i, $(q^n(i,j))_{n=0}^{\infty}$ alternates between $0, 1$. The chain is periodic.

To prove the Ergodic theorem, we use the strong Markov property.

Def. Let $\sigma_j^{(n)}$ be the n^{th} excursion time to state j:

$$\sigma_j^{(1)} := \inf\{n \geq 1 : X_n = j\} = \tau_j$$
$$\sigma_j^{(n)} := \inf\{n \geq 1 : X_{n+\sigma_j^{(n-1)}} = j\}$$

} The time it takes to return to j after the previous visit.

Lemma: Relative to $\mathbb{P}^i$ for any i, $\{\sigma_j^{(n)}\}_{n=1}^{\infty}$ are independent, and $\{\sigma_j^{(n)}\}_{n=2}^{\infty}$ are iid with

$$\mathrm{Law}_{\mathbb{P}^i}(\sigma_j^{(n)}) = \mathrm{Law}_{\mathbb{P}^j}(\tau_j) \quad \forall i, \ \forall n \geq 2.$$

Pf. Let $\tau_j^{(n)} = \inf\{n > \tau_j^{(n-1)} : X_n = j\}$, where $\tau_j^{(0)} = 0$.

$\tau_j^{(1)} = \tau_j$.

I.e. $\tau_j^{(n)}$ is the n^{th} passage time to j. stopping time.

Note that

$$\sigma_j^{(n+1)}(X_0, X_1, X_2, \ldots) = \sigma_j^{(1)}\left(X_{\tau_j^{(n)}}, X_{\tau_j^{(n)}+1}, X_{\tau_j^{(n)}+2}, \ldots\right)$$

$$\sigma_j^{(n+1)}(X_0, X_1, X_2, \dots) = \sigma_j^{(1)}\left(X_{\tau_j^{(n)}}, X_{\tau_j^{(n)}+1}, X_{\tau_j^{(n)}+2}, \dots\right)$$

Since $X_{\tau_j^{(n)}} = j$, and since $\tau_j^{(n)}$ is a stopping time, by the Strong Markov property:

- $\sigma_j^{(n+1)}$ is independent from $(X_0, X_1, \dots, X_{\tau_j^{(n)}})$ $\therefore \sigma_j^{(n+1)}$ indep from $\sigma_j^{(1)}, \dots, \sigma_j^{(n)}$.

  ↳ But $\sigma_j^{(1)}, \dots, \sigma_j^{(n)}$ are functions of

- As $X_{\tau_j^{(n)}} = j$, $(X_{\tau_j^{(n)}}, X_{\tau_j^{(n)}+1}, \dots) \overset{d}{=} (X_0, X_1, \dots)$ under $\mathbb{P}^j$.

$$\therefore \ \mathrm{Law}_{\mathbb{P}^i}\left(\sigma_j^{(n+1)}\right) = \mathrm{Law}_{\mathbb{P}^j}\big(\underbrace{\sigma_j^{(1)}}_{=\tau_j}\big) \quad \forall n \geq 1 \qquad ///$$

Pf. of Ergodic theorem: By SLLN:

$$\forall i, \quad \lim_{N\to\infty} \frac{1}{N}\left[\cancel{\sigma_j^{(1)}} + \sigma_j^{(2)} + \dots + \sigma_j^{(N)}\right]$$

$$= \mathbb{E}^i[\sigma_j^{(2)}] = \mathbb{E}^j[\tau_j] \qquad \mathbb{P}^i\text{-a.s.}$$

But $\sigma_j^{(1)} + \sigma_j^{(2)} + \dots + \sigma_j^{(N)} = \tau_j^{(N)}$ $\quad \therefore \lim_{N\to\infty} \frac{\tau_j^{(N)}}{N} = \mathbb{E}^j[\tau_j]$.

$$V_j(N) = \sum_{n=0}^{\infty} \mathbb{1}_{X_n = j} = \#\text{ visits to } j \text{ during first } N \text{ steps.}$$

$$\therefore \quad \tau_j^{V_j(N)} \le N \le \tau_j^{V_j(N)+1}$$

$$\frac{\tau_j^{V_j(N)}}{V_j(N)} \le \frac{N}{V_j(N)} \le \frac{\tau_j^{V_j(N)+1}}{V_j(N)+1} \cdot \frac{V_j(N)+1}{V_j(N)}$$

$$\frac{\tau_j^{V_j(N)}}{V_j(N)} \xrightarrow{N\to\infty} \mathbb{E}^j[\tau_j], \qquad \frac{\tau_j^{V_j(N)+1}}{V_j(N)+1} \xrightarrow{N\to\infty} \mathbb{E}^j[\tau_j], \qquad \frac{V_j(N)+1}{V_j(N)} \xrightarrow{N\to\infty} 1$$

$$\therefore \quad \frac{V_j(N)}{N} \to \frac{1}{\mathbb{E}^j[\tau_j]}.$$

///