We've seen several versions of the Markov property.
The most powerful form, for time-homogeneous processes, was in [Lec 38.2]

If X is a time homogeneous Markov process $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}) \to (S, \mathcal{B})$ and $\{\mathbb{P}^\nu : \nu \in \mathrm{Prob}(S,\mathcal{B})\}$ are the associated probability measures on $(S^T, \mathcal{B}^{\otimes T})$ with $\mathbb{P}^\nu(X_0 \in B) = \nu(B)$, the $\forall\, F \in \mathbb{B}(S^T, \mathcal{B}^{\otimes T})$,

$$\mathbb{E}^\nu[F(X_{t+\cdot}) \mid \mathcal{F}_t] = \mathbb{E}^x[F(X)]\Big|_{x=X_t}.$$

Theorem: Let $T=\mathbb{N}$. Let $\tau : (\Omega, \{\mathcal{F}_n\}_{n\in\mathbb{N}}) \to \mathbb{N}\cup\{\infty\}$ be a stopping time.
Then

$$\mathbb{E}^\nu[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] = \mathbb{E}^x[F(X)]\Big|_{x=X_\tau} \quad \mathbb{P}^\nu\text{-a.s.}$$

on $\{\tau<\infty\}$.

I.e. Conditioned on the value at the random time τ, the process restarts fresh.

Pf. We know how to condition on $\mathcal{F}_\tau$:

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] \mathbb{1}_{\tau<\infty} = \sum_{n=0}^{\infty} \mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_n] \mathbb{1}_{\{\tau=n\}}$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[F(X_{\cancel{\tau}\,n+\cdot}) \mathbb{1}_{\{\tau=n\}} \mid \mathcal{F}_n]$$

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}[F(X_{n+\cdot}) \mid \mathcal{F}_n]}_{\mathbb{E}^x[F(X)]\big|_{x=X_n}} \mathbb{1}_{\{\tau=n\}}$$

I.e. if $g(x) = \mathbb{E}^x[F(X)]$, then

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] \mathbb{1}_{\tau<\infty} = \sum_{n=0}^{\infty} g(X_n) \mathbb{1}_{\{\tau=n\}}$$

$$= g(X_\tau) \mathbb{1}_{\{\tau<\infty\}}$$

$$= \mathbb{E}^x[F(X)]\Big|_{x=X_\tau} \mathbb{1}_{\{\tau<\infty\}}$$

///.

To be clear, we can rephrase this (in the discrete spacetime context) as:

Cor: Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain in the discrete state space S.
Let τ be a stopping time (adapted to the same filtration as $(X_n)_{n\in\mathbb{N}}$).
For any $x\in S$, conditioned on $\{\tau<\infty, X_\tau = x\}$,

$\mathcal{F}_\tau$ and $\{X_{\tau+n}\}_{n\in\mathbb{N}}$ are independent, and

$(X_{\tau+n})_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ have the same distribution under $\mathbb{P}^x$.

Pf. Let $Y\in\mathbb{B}(\Omega,\mathcal{F}_\tau)$, $F\in\mathbb{B}(S^{\mathbb{N}},\mathcal{B}^{\otimes\mathbb{N}})$. Then for any initial distribution ν,

$$\mathbb{E}^\nu[F(X_{\tau+\cdot})\underbrace{Y\mathbb{1}_{\{\tau<\infty, X_\tau=x\}}}_{\mathcal{F}_\tau\text{-meas.}}]$$

$$= \mathbb{E}^\nu[\mathbb{E}[F(X_{\tau+\cdot})|\mathcal{F}_\tau]\,Y\mathbb{1}_{\{\tau<\infty, X_\tau=x\}}]$$

$$= \mathbb{E}^\nu[\mathbb{E}^{\cancel{X_\tau}\,x}[F(X)]\,Y\mathbb{1}_{\{\tau<\infty, X_\tau=x\}}]$$

$$= \mathbb{E}^x[F(X)]\,\mathbb{E}^\nu[Y\mathbb{1}_{\{\tau<\infty, X_\tau=x\}}].$$

Thus $\mathbb{E}^\nu[F(X_{\tau+\cdot})Y : \tau<\infty, X_\tau = x] = \mathbb{E}^x[F(X)]\,\mathbb{E}^\nu[Y : \tau<\infty, X_\tau = x]$.

In other words: $\mathbb{E}^\nu[F(X_{\tau+\cdot})Y \mid \tau<\infty, X_\tau = x]$

$$= \mathbb{E}^x[F(X)]\,\mathbb{E}^\nu[Y \mid \tau<\infty, X_\tau=x]. \qquad \star$$

In particular, taking $Y \equiv 1$ shows

$$\mathbb{E}^\nu[F(X_{\tau+\cdot}) \mid \tau<\infty, X_\tau=x] = \mathbb{E}^x[F(X)]. \qquad \star\star$$

I.e. conditioned on $\{\tau<\infty, X_\tau=x\}$, $(X_{\tau+\cdot}) \stackrel{d}{=} (X_\cdot)$ under $\mathbb{P}^x$.

Moreover, $\star + \star\star \Rightarrow \quad \mathbb{E}' = \mathbb{E}^\nu[\,\cdot \mid \tau<\infty, X_\tau=x]$

$$\mathbb{E}'[F(X_{\tau+\cdot})Y] = \mathbb{E}'[F(X_{\tau+\cdot})]\,\mathbb{E}'[Y].$$

This holds $\forall\, Y \in \mathcal{B}(\Omega, \mathcal{F}_\tau)$, and

$\therefore\ (X_{\tau+\cdot})$ is independent from $\mathcal{F}_\tau$. ///

conditioned on $\{\tau<\infty, X_\tau=x\}$

In particular: $(X_0, \dots, X_\tau), (X_\tau, X_{\tau+1}, X_{\tau+2}, \dots)$ are independent.