Conditioning on $\mathcal{F}_\tau$

$\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$ is a σ-subfield of $\mathcal{F}$, so we know how to make sense of $\mathbb{E}[X|\mathcal{F}_\tau]$.

- Averaging property
- Tower property e.g. if $\sigma \leq \tau$ then

In this discrete time setting, there is a simple expression for $\mathbb{E}_{\mathcal{F}_\tau}$.

Prop: Let τ be a stopping time, and let $X \in L^1$ or $X \geq 0$. Then

$$\mathbb{E}_{\mathcal{F}_\tau}[X] =$$

I.e. if $X_n := \mathbb{E}[X|\mathcal{F}_n]$ $n \in \mathbb{N} \cup \{\infty\}$, then $\mathbb{E}[X|\mathcal{F}_\tau] =$

Pf. We saw last time that X_τ is $\mathcal{F}_\tau$-measurable.

$$\sum_{n \leq \infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}} | X_n]]$$

So, if $X \in L^1$ then $\sum_{n \leq \infty} \mathbb{E}[\mathbb{1}_{\{\tau = n\}} |X_n|] \leq \mathbb{E}[|X|] < \infty$.

$$\therefore \mathbb{E}[|X_\tau|] =$$

Now, if $E \in \mathcal{F}_\tau$, then

$$\mathbb{E}[X \mathbb{1}_E] = \mathbb{E}[X \mathbb{1}_E \qquad]$$

$$= \mathbb{E}\Big[\sum_{n \leq \infty} \mathbb{1}_{\{\tau = n\}} X_n \cdot \mathbb{1}_E\Big] = \mathbb{E}[X_\tau \mathbb{1}_E]$$

We also have a handy reformulation of the tower property.

Prop: Let $X \in L^1$ or $X \geq 0$ be $\mathcal{F}$-measurable.
If σ, τ are any two stopping times, then

1. $\mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_\tau] = \mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$
2. $\mathbb{1}_{\tau > \sigma} \mathbb{E}[X \mid \mathcal{F}_\sigma] = \mathbb{1}_{\tau > \sigma} \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$
3. $\mathbb{E}_{\mathcal{F}_\sigma} \mathbb{E}_{\mathcal{F}_\tau}[X] = \mathbb{E}_{\mathcal{F}_\tau} \mathbb{E}_{\mathcal{F}_\sigma}[X] = \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$

(Note: $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \wedge \mathcal{F}_\tau$ [HW].)

Pf. 1. $\mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_\tau] = \sum_{n \leq \infty} \mathbb{1}_{\tau \leq \sigma} \mathbb{1}_{\tau = n} \mathbb{E}[X \mid \mathcal{F}_n]$

$$= \mathbb{1}_{\tau \leq \sigma} \sum_{n \leq \infty} \mathbb{1}_{\tau \wedge \sigma = n} \mathbb{E}[X \mid \mathcal{F}_n]$$

$$= \mathbb{1}_{\tau \leq n} \mathbb{E}[X \mid \mathcal{F}_{\tau \wedge \sigma}].$$

2. Very similar to 1.

3. Here, we use the fact that

$\{\tau \le \sigma\}$ and $\{\tau > \sigma\}$ are in $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ [HW]

$$\therefore\ \mathbb{E}_{\mathcal{F}_\sigma}\mathbb{E}_{\mathcal{F}_\tau}[X] = \mathbb{E}_{\mathcal{F}_\sigma}\left[(\mathbb{1}_{\tau\le\sigma} + \mathbb{1}_{\tau>\sigma})\mathbb{E}_{\mathcal{F}_\tau}[X]\right]$$

$$= \mathbb{1}_{\tau\le\sigma}\mathbb{E}_{\mathcal{F}_\sigma}\left[\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X]\right] + \mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_\sigma}\left[\mathbb{E}_{\mathcal{F}_\tau}[X]\right]$$

$$= \mathbb{1}_{\tau\le\sigma}\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X] + \mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X]$$