Conditioning on $\mathcal{F}_\tau$

$\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$ is a σ-subfield of $\mathcal{F}$, so we know how to make sense of $\mathbb{E}[X|\mathcal{F}_\tau]$.

- Averaging property $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}_{\mathcal{F}_\tau}[X]Y]$ $\forall Y \in B(\mathcal{F}_\tau)$. ($\mathcal{F}_\tau$-meas.)
- Tower property e.g. if $\sigma \leq \tau$ then $\mathbb{E}_{\mathcal{F}_\tau}\mathbb{E}_{\mathcal{F}_\sigma} = \mathbb{E}_{\mathcal{F}_\sigma}\mathbb{E}_{\mathcal{F}_\tau} = \mathbb{E}_{\mathcal{F}_\sigma}$.

In this discrete time setting, there is a simple expression for $\mathbb{E}_{\mathcal{F}_\tau}$.

Prop: Let τ be a stopping time, and let $X \in L^1$ or $X \geq 0$. Then

$$\mathbb{E}_{\mathcal{F}_\tau}[X] = \sum_{n \leq \infty} \mathbb{1}_{\{\tau = n\}} \mathbb{E}[X|\mathcal{F}_n]$$

I.e. if $X_n := \mathbb{E}[X|\mathcal{F}_n]$ $n \in \mathbb{N} \cup \{\infty\}$, then $\mathbb{E}[X|\mathcal{F}_\tau] = X_\tau$

Pf. We saw last time that X_τ is $\mathcal{F}_\tau$-measurable.

$$\sum_{n \leq \infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}}|X_n|] \leq \sum_{n \leq \infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}} \mathbb{E}_{\mathcal{F}_n}[|X|]]$$

$$= \sum_{n \leq \infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}}|X|]$$

$$= \mathbb{E}[|X|] < \infty.$$

So, if $X \in L^1$ then $\sum_{n \le \infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}} |X_n|] \le \mathbb{E}[|X|] < \infty$.

$$\therefore \mathbb{E}[|X_\tau|] = \mathbb{E}\left[\left|\sum_{n\le\infty} \mathbb{1}_{\{\tau=n\}} X_n\right|\right] \le \quad \therefore \le \mathbb{E}[|X|] < \infty.$$

Now, if $E \in \mathcal{F}_\tau$, then

$$\mathbb{E}[X \mathbb{1}_E] = \mathbb{E}[X \mathbb{1}_E \sum_{n\le\infty} \mathbb{1}_{\{\tau=n\}}]$$

$$= \sum_{n\le\infty} \mathbb{E}[X \mathbb{1}_{E\cap\{\tau=n\}}]$$

$E \in \mathcal{F}_\tau$

$\therefore E \cap \{\tau = n\} \in \mathcal{F}_n$

$= \mathbb{E}[\underbrace{\mathbb{E}_{\mathcal{F}_n}[X]}_{X_n} \mathbb{1}_{E\cap\{\tau=n\}}]$

$$= \sum_{n\le\infty} \mathbb{E}[X_n \mathbb{1}_{E\cap\{\tau=n\}}]$$

$$= \sum_{n\le\infty} \mathbb{E}[\mathbb{1}_{\{\tau=n\}} X_n \mathbb{1}_E]$$

$$= \mathbb{E}\left[\sum_{n\le\infty} \mathbb{1}_{\{\tau=n\}} X_n \cdot \mathbb{1}_E\right] = \mathbb{E}[X_\tau \mathbb{1}_E]$$

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$\mathbb{E}[X|\mathcal{F}_\tau] = X_\tau$

$X_n = \mathbb{E}[X|\mathcal{F}_n]$

We also have a handy reformulation of the tower property.

Prop: Let $X \in L^1$ or $X \geq 0$ be $\mathcal{F}$-measurable.
If σ, τ are any two stopping times, then

1. $\mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_\tau] = \mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$
2. $\mathbb{1}_{\tau > \sigma} \mathbb{E}[X \mid \mathcal{F}_\sigma] = \mathbb{1}_{\tau > \sigma} \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$
3. $\mathbb{E}_{\mathcal{F}_\sigma} \mathbb{E}_{\mathcal{F}_\tau}[X] = \mathbb{E}_{\mathcal{F}_\tau} \mathbb{E}_{\mathcal{F}_\sigma}[X] = \mathbb{E}[X \mid \mathcal{F}_{\sigma \wedge \tau}]$

does not require $\sigma \leq \tau$ or $\tau \leq \sigma$.

$\mathbb{E}_{\mathcal{G}} \mathbb{E}_{\mathcal{H}} X \neq \mathbb{E}_{\mathcal{G} \cap \mathcal{H}}$

(Note: $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \wedge \mathcal{F}_\tau$ [HW].)

Pf. 1. $\mathbb{1}_{\tau \leq \sigma} \mathbb{E}[X \mid \mathcal{F}_\tau] = \sum_{n \leq \infty} \underbrace{\mathbb{1}_{\tau \leq \sigma} \mathbb{1}_{\tau = n}}_{} \mathbb{E}[X \mid \mathcal{F}_n]$

$= 1$ if $\tau = n$ & $\tau \leq \sigma$, 0 otherwise

$= \mathbb{1}_{\{\tau \wedge \sigma = n,\ \tau \leq \sigma\}}$

$= \mathbb{1}_{\tau \leq \sigma} \sum_{n \leq \infty} \mathbb{1}_{\tau \wedge \sigma = n} \mathbb{E}[X \mid \mathcal{F}_n]$

$= \mathbb{1}_{\tau \leq n} \mathbb{E}[X \mid \mathcal{F}_{\tau \wedge \sigma}]$.

2. Very similar to 1.

3. Here, we use the fact that

$\{\tau \le \sigma\}$ and $\{\tau > \sigma\}$ are in $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ [HW]

$$\therefore\ \mathbb{E}_{\mathcal{F}_\sigma}\mathbb{E}_{\mathcal{F}_\tau}[X] = \mathbb{E}_{\mathcal{F}_\sigma}\left[(\mathbb{1}_{\tau\le\sigma}+\mathbb{1}_{\tau>\sigma})\mathbb{E}_{\mathcal{F}_\tau}[X]\right]$$

$$= \mathbb{E}_{\mathcal{F}_\sigma}\Big[\underbrace{\mathbb{1}_{\tau\le\sigma}\mathbb{E}_{\mathcal{F}_\tau}[X]}_{=\mathbb{1}_{\tau\le\sigma}\mathbb{E}_{\mathcal{F}_{\tau\wedge\sigma}}[X] \text{ by 1.}}\Big] + \mathbb{E}_{\mathcal{F}_\sigma}\left[\mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_\tau}[X]\right]$$

$$\in \mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_\sigma$$

$$= \mathbb{1}_{\tau\le\sigma}\underbrace{\mathbb{E}_{\mathcal{F}_\sigma}\left[\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X]\right]}_{\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X]} + \underbrace{\mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_\sigma}}\left[\mathbb{E}_{\mathcal{F}_\tau}[X]\right]$$

$$\mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}\Big[\underbrace{\mathbb{E}_{\mathcal{F}_\tau}[X]\Big]}_{=\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X]}$$

$$= \mathbb{1}_{\tau\le\sigma}\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X] + \mathbb{1}_{\tau>\sigma}\mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X] = \mathbb{E}[X \mid \mathcal{F}_{\sigma\wedge\tau}].$$

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