If $X_n : (\Omega, \mathcal{F}) \to (S, \mathcal{B})$ are rv's for $n \in \mathbb{N} \cup \{\infty\}$, and $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is a random time, then

X_τ is a random variable.

Now, if $(X_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is adapted to $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ we'd like to say X_τ is "$\mathcal{F}_\tau$" measurable.

What should this mean? Don't want a random σ-field!

To figure this out, note that

$$\{X_\tau \in B\} =$$

In particular, for any n, $\{\tau = n\} \cap \{X_\tau \in B\}$

$$= \{\tau = n\} \cap \{X_n \in B\}$$

Assume τ is a stopping time

Def. If τ is a stopping time, then

$$\mathcal{F}_\tau := \{E \subseteq \Omega : \{\tau = n\} \cap E \in \mathcal{F}_n \ \forall n \leq \infty\}$$

$$\mathcal{F}_\tau := \{E \subseteq \Omega : \{\tau = n\} \cap E \in \mathcal{F}_n \ \forall n \leq \infty\}.$$

Prop: If τ is a stopping time, then $\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$ is a σ-field, and τ is $\mathcal{F}_\tau$-measurable. Moreover, if $\sigma \leq \tau$ are stopping times, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$

Pf. $\Omega \cap \{\tau = n\} = \{\tau = n\} \in \mathcal{F}_n \ \forall n$ so $\Omega \in \mathcal{F}_\tau$.

If $E \in \mathcal{F}_\tau$, $E^c \cap \{\tau = n\}$

If $\{E_k\}_{k=1}^\infty \subseteq \mathcal{F}_\tau$, $\{\tau = n\} \cap \bigcap_{k=1}^\infty E_k$

Thus $\mathcal{F}_\tau$ is a σ-field.

Now, $\{\tau = n\} \cap \{\tau = k\} = \begin{cases} \emptyset & k \neq n \\ \{\tau = n\} & k = n \end{cases}$

$\therefore \{\tau = k\} \in \mathcal{F}_\tau \quad \forall k$

Finally, if $\sigma \leq \tau$ and $E \in \mathcal{F}_\sigma$, $E \cap \{\tau \leq n\}$

What does $\mathcal{F}_\tau$-measurability mean?

Prop: Let τ be a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}})$ and let $Z: \Omega \to \mathbb{R}$.

TFAE: 1. Z is $\mathcal{F}_\tau$-measurable.

2. $\mathbb{1}_{\{\tau \le n\}} Z$ is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N} \cup \{\infty\}$

3. $\mathbb{1}_{\{\tau = n\}} Z$ is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N} \cup \{\infty\}$

4. $Z = Y_\tau$ for some adapted $\mathbb{R}$-valued stochastic process $\{Y_n\}_{n\in\mathbb{N}\cup\{\infty\}}$.

Pf. $(1 \Rightarrow 2)$ Since Z is $\mathcal{F}_\tau$-measurable, $\{Z \in B\} \cap \{\tau \le n\} \in \mathcal{F}_n$ $\forall n$, $B \in \mathcal{B}(\mathbb{R})$.

- S'pose $0 \notin B$. $\{\mathbb{1}_{\{\tau\le n\}} Z \in B\}$
- OTOH, $\{\mathbb{1}_{\{\tau \le n\}} Z = 0\}$

$(2 \Rightarrow 3)$ $\mathbb{1}_{\{\tau = n\}} = \mathbb{1}_{\{\tau \le n\}} - \mathbb{1}_{\{\tau \le n-1\}}$

$(3 \Rightarrow 4)$ Define $Y_n :=$

$Y_\tau(\omega) =$

$(4 \Rightarrow 1)$ we have left to prove that if $Y_n : \Omega \to \mathbb{R}$ is adapted (including Y_∞), then Y_τ is $\mathcal{F}_\tau$-measurable. To that end, note

$$Y_\tau =$$

So, need to show that if W is $\mathcal{F}_k$-measurable, then $\mathbb{1}_{\{\tau=k\}} W$ is $\mathcal{F}_\tau$-meas.
Suffices to prove this in the special case $W = \mathbb{1}_E$ for any $E \in \mathcal{F}_k$

$$W \mathbb{1}_{\{\tau=k\}} = \mathbb{1}_E \mathbb{1}_{\{\tau=k\}}$$

So we need only check that

Cor: If $(X_n)_{n \in \mathbb{N}}$ is an adapted process in $(S, \mathcal{B})$ and τ is a finite stopping time, then X_τ is $\mathcal{F}_\tau / \mathcal{B}$-meas.

Pf. $\forall B \in \mathcal{B}$, $X_\tau^{-1}(B) =$