If $X_n : (\Omega, \mathcal{F}) \to (S, \mathcal{B})$ are rv's for $n \in \mathbb{N} \cup \{\infty\}$, and $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is a random time, then

measurable (proved on 280A final).

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) \quad \text{is a random variable.}$$

Now, if $(X_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is adapted to $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ (X_∞ is $\mathcal{F}_\infty / \mathcal{B}$-meas.) we'd like to say X_τ is "$\mathcal{F}_\tau$" measurable.

What should this mean? Don't want a random σ-field!

To figure this out, note that

$$\{X_\tau \in B\} = \bigcup_{n \le \infty} \{X_n \in B, \tau = n\}$$

In particular, for any n, $\{\tau = n\} \cap \{X_\tau \in B\}$

$$= \underbrace{\{\tau = n\}}_{\in \mathcal{F}_n} \cap \underbrace{\{X_n \in B\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n.$$

Assume τ is a stopping time

Def. If τ is a stopping time, then

$$\mathcal{F}_\tau := \{E \subseteq \Omega : \{\tau = n\} \cap E \in \mathcal{F}_n \ \forall n \le \infty\} = \mathcal{F}_k$$

E.g. $\tau \equiv k$. $\quad \{\tau = n\} \cap E = \begin{cases} E & n = k \\ \emptyset & n \ne k \end{cases}$

$$\mathcal{F}_\tau := \{E \subseteq \Omega : \{\tau = n\} \cap E \in \mathcal{F}_n \ \forall n \leq \infty\}.$$

$\{\tau \leq n\}$ (Exercise)

Prop: If τ is a stopping time, then $\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$ is a σ-field, and τ is $\mathcal{F}_\tau$-measurable. Moreover, if $\sigma \leq \tau$ are stopping times, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$

Pf. $\Omega \cap \{\tau = n\} = \{\tau = n\} \in \mathcal{F}_n \ \forall n$ so $\Omega \in \mathcal{F}_\tau$.

If $E \in \mathcal{F}_\tau$, $E^c \cap \{\tau = n\} = \{\tau = n\} \setminus E = \underbrace{\{\tau = n\}}_{\in \mathcal{F}_n} \setminus (\underbrace{E \cap \{\tau = n\}}_{\in \mathcal{F}_n}) \in \mathcal{F}_n \quad \therefore E^c \in \mathcal{F}_\tau$.

If $\{E_k\}_{k=1}^\infty \subseteq \mathcal{F}_\tau$, $\{\tau = n\} \cap \bigcap_{k=1}^\infty E_k$

$$= \bigcap_{k=1}^\infty \underbrace{\{\tau = n\} \cap E_k}_{\mathcal{F}_n} \in \mathcal{F}_n.$$

Thus $\mathcal{F}_\tau$ is a σ-field.

$\tau \leq \infty$.

Now, $\{\tau = n\} \cap \{\tau = k\} = \begin{cases} \emptyset & k \neq n \\ \{\tau = n\} & k = n \end{cases} \in \mathcal{F}_n. \ \forall n.$

$\therefore \{\tau = k\} \in \mathcal{F}_\tau \ \forall k \quad \therefore \tau$ is $\mathcal{F}_\tau$-meas.

Finally, if $\sigma \leq \tau$ and $E \in \mathcal{F}_\sigma$, $E \cap \{\tau \leq n\}$

$\Downarrow$

$\{\tau \leq n\} \subseteq \{\sigma \leq n\}$

$$= (\underbrace{E \cap \{\sigma \leq n\}}_{\in \mathcal{F}_n}) \cap \underbrace{\{\sigma \leq n\}}_{\mathcal{F}_n} \therefore \in \mathcal{F}_n$$

$\therefore E \in \mathcal{F}_\tau$. ///

What does $\mathcal{F}_\tau$ - measurability mean?

Prop: Let τ be a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}})$ and let $Z : \Omega \to \mathbb{R}$.

TFAE: 1. Z is $\mathcal{F}_\tau$ - measurable.

2. $\mathbb{1}_{\{\tau \le n\}} Z$ is $\mathcal{F}_n$ - measurable $\forall n \in \mathbb{N} \cup \{\infty\}$

3. $\mathbb{1}_{\{\tau = n\}} Z$ is $\mathcal{F}_n$ - measurable $\forall n \in \mathbb{N} \cup \{\infty\}$

4. $Z = Y_\tau$ for some adapted $\mathbb{R}$-valued stochastic process $\{Y_n\}_{n \in \mathbb{N} \cup \{\infty\}}$.

Pf. $(1 \Rightarrow 2)$ Since Z is $\mathcal{F}_\tau$ - measurable, $\{Z \in B\} \cap \{\tau \le n\} \in \mathcal{F}_n$ $\forall n$, $B \in \mathcal{B}(\mathbb{R})$.

- S'pose $0 \notin B$. $\{\mathbb{1}_{\{\tau \le n\}} Z \in B\} = \{Z \in B\} \cap \{\tau \le n\} \in \mathcal{F}_n$.
- OTOH, $\{\mathbb{1}_{\{\tau \le n\}} Z = 0\}^c = \{Z \neq 0\} \cap \{\tau \le n\} \in \mathcal{F}_n$

$\therefore$ $\mathbb{1}_{\{\tau \le n\}} Z$ is $\mathcal{F}_n$ - mbs.

$(2 \Rightarrow 3)$ $\mathbb{1}_{\{\tau = n\}} Z = \mathbb{1}_{\{\tau \le n\}} Z - \mathbb{1}_{\{\tau \le n-1\}} Z$

$(3 \Rightarrow 4)$ Define $Y_n := \mathbb{1}_{\{\tau = n\}} Z$. adapted.

$$Y_\tau(\omega) = \mathbb{1}_{\{\omega : \tau(\omega) = \tau(\omega)\}} Z(\omega) = Z(\omega).$$

$(4 \Rightarrow 1)$ we have left to prove that if $Y_n : \Omega \to \mathbb{R}$ is adapted (including Y_∞), then Y_τ is $\mathcal{F}_\tau$-measurable. To that end, note

$$Y_\tau = \sum_{k \leq \infty} \mathbb{1}_{\{\tau = k\}} Y_k \quad \therefore \text{ suffices to show } \mathbb{1}_{\{\tau=k\}} Y_k \text{ is } \mathcal{F}_\tau\text{-meas. } \forall k.$$

So, need to show that if W is $\mathcal{F}_k$-measurable, then $\mathbb{1}_{\{\tau=k\}} W$ is $\mathcal{F}_\tau$-meas. Suffices to prove this in the special case $W = \mathbb{1}_E$ for any $E \in \mathcal{F}_k$ by Dynkin.

$$W \mathbb{1}_{\{\tau=k\}} = \mathbb{1}_E \mathbb{1}_{\{\tau=k\}} = \mathbb{1}_{E \cap \{\tau=k\}}$$

So we need only check that $E \cap \{\tau = k\} \in \mathcal{F}_\tau$.

$$(E \cap \{\tau=k\}) \cap \{\tau=n\} = \begin{cases} \emptyset & k \neq n \\ E \cap \{\tau=n\} & k = n \end{cases} \in \mathcal{F}_n \quad \forall n.$$

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<u>Cor</u>: If $(X_n)_{n \in \mathbb{N}}$ is an adapted process in $(S, \mathcal{B})$ and τ is a finite stopping time, then X_τ is $\mathcal{F}_\tau / \mathcal{B}$-meas.

<u>Pf.</u> $\forall B \in \mathcal{B}$, $X_\tau^{-1}(B) = (\underbrace{\mathbb{1}_B \circ X_\tau}_{Y_\tau})^{-1}(1) \in \mathcal{F}_\tau$

$Y_n = \mathbb{1}_B \circ X_n : \Omega \to \mathbb{R}$ adapted.

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