Stopping Times

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space.
A random variable $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ is a (discrete) stopping time iff

$$\{\tau \leq n\} \in \mathcal{F}_n \qquad \forall n \in \mathbb{N}.$$

Equivalently: iff the process

Eg. If $(X_n)_{n\in\mathbb{N}}$ is an adapted process in $(S, \mathcal{B})$ and $B \in \mathcal{B}$, then $T_B = \inf\{n \geq 0 : X_n \in B\}$ is a stopping time.

$$\{T_B \leq n\}$$

How about the final hitting time $L_B = \sup\{n \geq 0 : X_n \in B\}$?

$$\{L_B \leq n\} = \bigcup_{k \leq n} \{L_B = k\}$$

Lemma: TFAE:

1. τ is a stopping time, i.e. $\{\tau \le n\} \in \mathcal{F}_n \ \forall n \in \mathbb{N}$.
2. $\{\tau > n\} \in \mathcal{F}_n \ \forall n \in \mathbb{N}$.
3. $\{\tau = n\} \in \mathcal{F}_n \ \forall n \in \mathbb{N}$.

Moreover, if any one of these conditions hold, then they also hold for $n = \infty$.

Pf. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow readily from the identities

$$\{\tau > n\}^c = \{\tau \le n\} = \bigcup_{k=0}^{n} \{\tau = k\}, \quad \{\tau = n\} = \{\tau \le n\} \setminus \{\tau \le n-1\}$$

Now, if τ is a stopping time, then

$$\{\tau < \infty\} =$$

$$\therefore \quad \{\tau = \infty\} =$$

Clearly $\{\tau \le \infty\}$ and $\{\tau > \infty\}$

Eg. We saw the first hitting time of an adapted process is a stopping time. How about the second hitting time? The billionth?

Lemma: If $(X_n)_{n\in\mathbb{N}}$ is an adapted process in $(S,\mathcal{B})$, $B\in\mathcal{B}$, and σ is a stopping time, then $\tau=\inf\{n>\sigma : X_n\in B\}$ is a stopping time.

Pf. $\{\tau=n\} = \bigcup_{k\in\mathbb{N}\cup\{\infty\}} \{\tau=n, \sigma=k\}$

We can combine stopping times to make new ones.

Lemma: If $\sigma, \tau, \{\tau_k\}_{k=1}^{\infty}$ are stopping times, then

1. $\sigma\wedge\tau$, $\sigma\vee\tau$, $\sigma+\tau$ are stopping times.
2. If $\{\tau_k\}_{k=1}^{\infty}$ is monotone, then $\lim_{k\to\infty}\tau_k$ is a stopping time.

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Pf. 1. $\{\sigma \wedge \tau \leq n\}$ $\qquad$ $\{\sigma \vee \tau \leq n\}$

$\{\sigma + \tau = n\}$

2. S'pose $\tau_k \uparrow \tau_\infty$. We already know τ_∞ is measurable.

$\{\tau_\infty \leq n\}$ iff

The argument for $\tau_k \downarrow \tau_\infty$ is similar

Cor: If $\{\tau_k\}_{k=1}^{\infty}$ are stopping times, then so are

$\sup_k \tau_k$ $\qquad$ $\inf_k \tau_k$

$\limsup_k \tau_k$ $\qquad$ $\liminf_k \tau_k$