Stopping Times

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space.

A random variable $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ is a (discrete) stopping time iff

$$\{\tau \leq n\} \in \mathcal{F}_n \qquad \forall n \in \mathbb{N}.$$

Equivalently: iff the process $(\mathbb{1}_{\tau \leq n})_{n\in\mathbb{N}}$ is adapted.

E.g. If $(X_n)_{n\in\mathbb{N}}$ is an adapted process in $(S, \mathcal{B})$ and $B \in \mathcal{B}$, then $T_B = \inf\{n \geq 0 : X_n \in B\}$ is a stopping time.

$$\{T_B \leq n\} = \{\exists_n k \leq n \text{ s.t. } X_k \in B\}$$
$$= \bigcup_{k=0}^{n} \underbrace{\{X_k \in B\}}_{\in \mathcal{F}_k} \in \mathcal{F}_n.$$

How about the final hitting time $L_B = \sup\{n \geq 0 : X_n \in B\}$?

$$\{L_B \leq n\} = \bigcup_{k \leq n} \{L_B = k\} \qquad \in \sigma(X_k, X_{k+1}, \ldots)$$

Not a stopping time. $\qquad \{X_k \in B, X_{k+1} \notin B, X_{k+2} \notin B, \ldots\}$

Lemma: TFAE:

1. τ is a stopping time, i.e. $\{\tau \le n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$.
2. $\{\tau \ge n+1\} = \{\tau > n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$.
3. $\{\tau = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$.

$$\mathcal{F}_\infty = \sigma\Big(\bigcup_{k \in \mathbb{N}} \mathcal{F}_k\Big)$$

Moreover, if any one of these conditions hold, then they also hold for $n = \infty$.

Pf. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow readily from the identities

$$\{\tau > n\}^c = \{\tau \le n\} = \bigcup_{k=0}^{n} \{\tau = k\}, \quad \{\tau = n\} = \{\tau \le n\} \setminus \{\tau \le n-1\}$$

Now, if τ is a stopping time, then

$$\{\tau < \infty\} = \bigcup_{n=1}^{\infty} \{\tau \le n\} \in \mathcal{F}_\infty .$$

$$\therefore \quad \{\tau = \infty\} = \{\tau < \infty\}^c \in \mathcal{F}_\infty .$$

Clearly $\{\tau \le \infty\} = \Omega \in \mathcal{F}_\infty$ and $\{\tau > \infty\} = \emptyset \in \mathcal{F}_\infty$. ///

Eg. We saw the first hitting time of an adapted process is a stopping time. How about the second hitting time? The billionth?

Lemma: If $(X_n)_{n\in\mathbb{N}}$ is an adapted process in $(S, \mathcal{B})$, $B \in \mathcal{B}$, and σ is a stopping time, then $\tau = \inf\{n > \sigma : X_n \in B\}$ is a stopping time.

Pf. $\{\tau = n\} = \bigcup_{k\in\mathbb{N}\cup\{\infty\}} \{\tau = n, \sigma = k\}$ if $\tau = n$, $\therefore$ $\sigma < n$.

$$= \bigcup_{k=0}^{n-1} \{\tau = n, \sigma = k\} = \bigcup_{k=0}^{n-1} \underset{\in\mathcal{F}_k}{\{\sigma = k\}} \cap \{\underset{\in\mathcal{F}_{k+1}}{X_{k+1} \in B^c}, \underset{\in\mathcal{F}_{k+2}}{X_{k+2} \in B^c}, \dots, \underset{\in\mathcal{F}_{n-1}}{X_{n-1} \in B^c}, \underset{\in\mathcal{F}_n}{X_n \in B}\} \quad \therefore \in \mathcal{F}_n \quad ///$$

We can combine stopping times to make new ones.

Lemma: If $\sigma, \tau, \{\tau_k\}_{k=1}^{\infty}$ are stopping times, then

1. $\sigma \wedge \tau$, $\sigma \vee \tau$, $\sigma + \tau$ are stopping times.
2. If $\{\tau_k\}_{k=1}^{\infty}$ is monotone, then $\lim_{k\to\infty} \tau_k$ is a stopping time.

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Pf. 1. $\{\sigma \wedge \tau \leq n\} = \{\sigma \leq n\} \cup \{\tau \leq n\} \in \mathcal{F}_n$. $\{\sigma \vee \tau \leq n\} = \{\sigma \leq n\} \cap \{\tau \leq n\} \in \mathcal{F}_n$.

$$\{\sigma + \tau = n\} = \bigcup_k \{\sigma + \tau = n, \sigma = k\} = \bigcup_{k=0}^{n} \underbrace{\{\sigma = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_n} \cap \underbrace{\{\tau = n-k\}}_{\in \mathcal{F}_{n-k} \subseteq \mathcal{F}_n} \in \mathcal{F}_n.$$

2. S'pose $\tau_k \uparrow \tau_\infty$. We already know τ_∞ is measurable.
$\{\tau_\infty \leq n\}$ iff $\tau_k \leq n$ $\forall k$.
$= \bigcap\limits_{k=1}^{\infty} \{\tau_k \leq n\} \in \mathcal{F}_n$.
The argument for $\tau_k \downarrow \tau_\infty$ is similar ///

Cor: If $\{\tau_k\}_{k=1}^{\infty}$ are stopping times, then so are

$$\sup_k \tau_k = \lim_{k\to\infty} \max\{\tau_1, \dots, \tau_k\} \qquad \inf_k \tau_k = \lim_{k\to\infty} \min\{\tau_1, \dots, \tau_k\}.$$

$$\limsup_k \tau_k = \lim_{k\to\infty} \sup_{n \geq k} \tau_n \qquad \liminf_k \tau_k = \lim_{k\to\infty} \inf_{n \geq k} \tau_n \quad ///$$