Let $(X_t)_{t\in T}$ be a Markov process with (homogeneous) transition semigroup $(q_t)_{t\in T}$. If the initial distribution $\mathrm{Law}(X_0)$ is μ, then for $t>0$,

$$\mathrm{Law}(X_t)$$

Def. A law μ is called invariant or stationary if

(By the Markov property: if μ is an invariant law and $\mathrm{Law}(X_{t_0})=\mu$ for some $t_0\in T$, then $\mathrm{Law}(\mu_t)$ $\forall t\geq t_0$.)

In the discrete-time setting, we only need to check the 1-step transition:

$$\mathrm{Law}(X_{n+1})(dy) = \int \mathrm{Law}(X_n)(dx)\, q(x,dy)$$

If the state space is discrete, this becomes a linear algebra question:

Let's focus on the finite state space, discrete-time setting, $S = \{1, \dots, d\}$

$$\mu(j) = \sum_i \mu(i) q(i,j)$$

$$\underline{v} = \begin{bmatrix} \mu(1) \\ \vdots \\ \mu(d) \end{bmatrix} \qquad P_{ij} = q(i,j)$$

Note: the row sums of P are 1. Thus

$$P\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \qquad \text{so} \quad (P-I)\begin{bmatrix}1\\\vdots\\1\end{bmatrix} = 0$$

We need a probability vector: $v_j \geq 0$, $\sum_j v_j = 1$.

So long as $w_j \geq 0$, and $\underline{w} \neq \underline{0}$ we can renormalize

Prop: Any finite state Markov chain has an invariant distribution.

Pf. We saw on the last slide that the transition matrix P possesses a vector $\underline{w} \neq \underline{0}$ s.t. $\underline{w}^T P = \underline{w}^T$. Define $\underline{v}$ by $\underline{v}_j :=$

Claim: $\underline{v}^T P = \underline{v}^T$.

$\hookrightarrow$ $v_j = |w_j| = \left| \sum_i w_i P_{ij} \right|$

So: if $\underline{v}^T P \neq \underline{v}^T$, it follows that there is some j for which

$\therefore$ $\sum_j v_j$

Thus, indeed, $\underline{v}^T P = \underline{v}^T$. Since $\underline{v} \neq \underline{0}$ and $\underline{v}_j \geq 0 \; \forall j$,

$$\mu_j := v_j / \sum_i v_i$$

defines a probability vector, and

$$\mu^T P =$$

(This is a version of the Perron-Fröbenius theorem.)

Note: the existence of an invariant distribution is not guaranteed in infinite state spaces.

Eg. Simple random walk on $\mathbb{Z}$ $\quad q(i,j) = \frac{1}{2}\mathbb{1}_{j=i+1} + \frac{1}{2}\mathbb{1}_{j=i-1}$

If μ satisfies $\mu(j) = \sum_i \mu(i) q(i,j)$

$\therefore \exists A, B \in \mathbb{R}$ s.t. $\mu(j) = A + Bj$.

- Since $\mu(j) \geq 0$ $\forall j \in \mathbb{Z}$,
- Since $\mu(j)$ must satisfy $\sum_{j \in \mathbb{Z}} \mu(j) = 1$,

In fact, the random walk does have an

invariant measure $\quad \mu(j) = 1 \quad \forall j$

and it is unique up to positive multiple. But this Markov chain does not have an invariant probability distribution.

Prop: Let $(X_n)_{n\in\mathbb{N}}$ be a finite state, irreducible Markov chain.
Then there is a unique invariant probability distribution μ:

$$\mu_i =$$

Pf. By first-step analysis

$$\mathbb{E}^i[\tau_j] = \sum_k q(i,k)\,\mathbb{E}^k[\tau_j(i,X)]$$

$$= 1 + \sum_{k\neq j} q(i,k)\,\mathbb{E}^k[\tau_j].$$

$\therefore$ if μ is an invariant distribution, then $\forall j$

$$\sum_i \mu_i \mathbb{E}^i[\tau_j] = \sum_i \mu_i + \sum_i \mu_i \sum_{k\neq j} q(i,k)\,\mathbb{E}^k[\tau_j]$$

We've shown that if μ is an invariant probability distribution, then

$$\mu_i \cdot \mathbb{E}^i[\tau_i] = 1.$$

we already proved that an invariant distribution exists; $\therefore$ this is it. ///

Cor: In the finite state irreducible setting, the unique invariant distribution is strictly positive.

Observation: In [Lec. 44.1] we proved that $0 < \mathbb{E}^i[\tau_i] < \infty \ \forall i$ (in the finite state irreducible case); we now have the additional fact:

A similar result holds for infinite chains: every irreducible, recurrent chain possesses a unique (up to scale) invariant measure, that is strictly positive.

Note: the equation $\mathbb{E}^i[\tau_j] = 1 + \sum_{k \neq j} q(i,k)\, \mathbb{E}^k[\tau_j]$

says that, if $\underline{u}^{(j)}$ is the vector $u_i^{(j)} =$, then

I.e. $\underline{u}^{(j)} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + P^{(j)} \underline{u}^{(j)}$

This can be used to compute expected passage times, with linear algebra.

$$\begin{bmatrix} \mathbb{E}^1[\tau_j] \\ \vdots \\ \mathbb{E}^d[\tau_j] \end{bmatrix} = \underline{u}^{(j)}$$

See [Driver, § 22.9.2] for many worked examples.