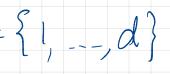
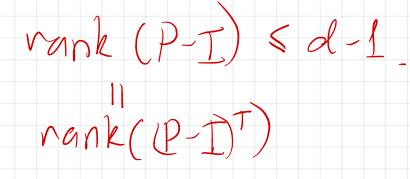
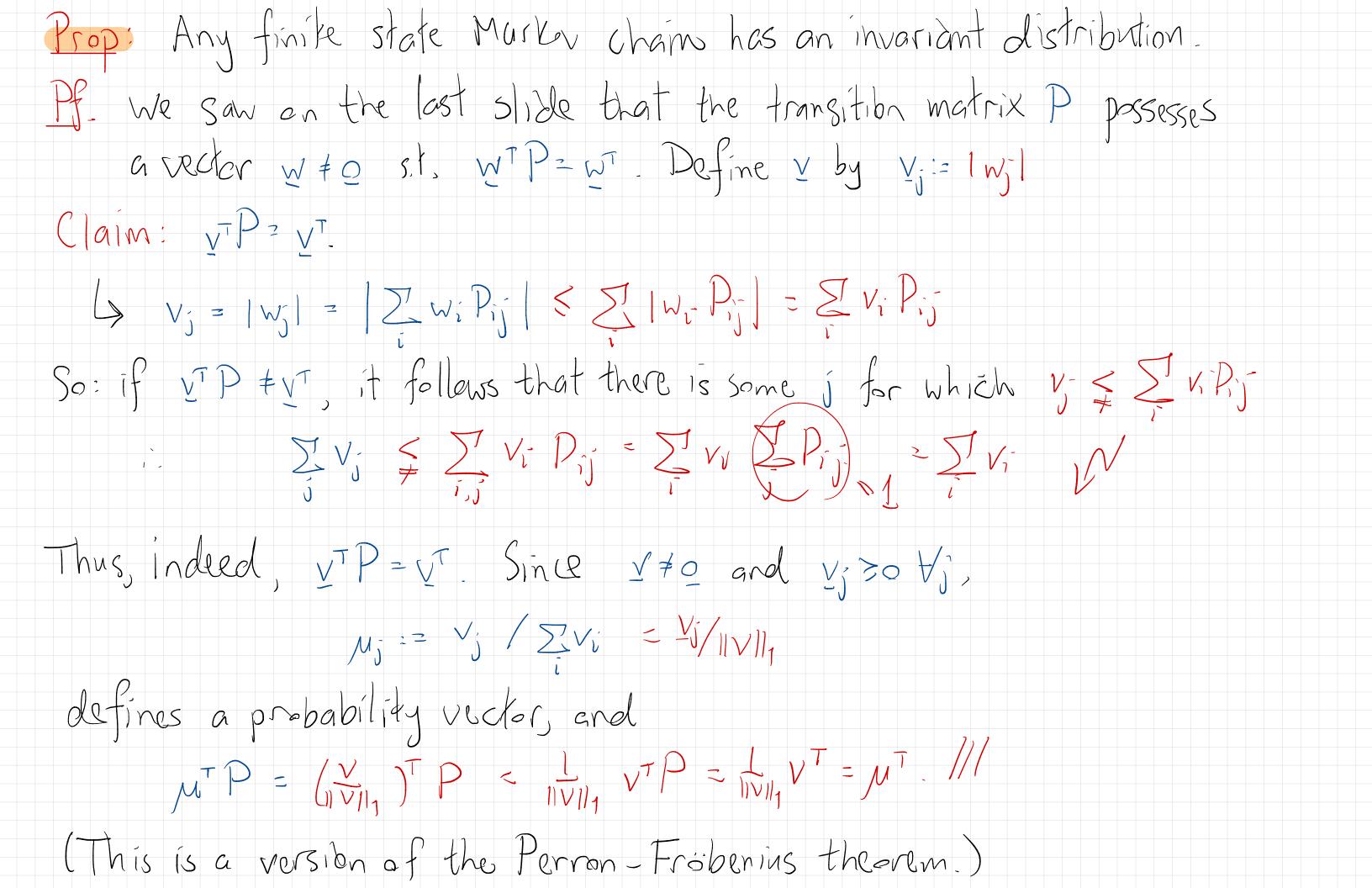
Let $(X_t)_{t\in T}$ be a Markov process with (homogeneous) transition semigroup $(q_t)_{t\in T}$. If the initial distribution $Law(X_0)$ is μ , then for t>0, $Law(X_t)(dy) = \int \mu(dx) q_t(y, dy) = \mu(dy) \forall teT$ Det A law mis called invariant or stationary if) (By the Markov property: if μ is an invariant law and $Law(X_{t_o}) = \mu$ for some locT, then $Law(\mu_t)(dy) = (\mu_t dy)(q_{t-t_o}(x, dy) = \mu_t dy)$. $\forall t \ge t_{o}$.) In the discrete time setting, we only need to check the 1-step transition: $Law(X_{n+1})(dy) = \int Law(X_n)(dy)q(y,dy)$ $= \int \mu(dy) q(w, dy) \frac{1}{2} strong \\ = \mu(dy). \int \mu d\mu(dy).$ If the state space is discrete, this becomes a linear algebra question: $\mu(j) = Z_{\mu}(i)q_{ij}$

Let's focus on the finite state space, discrete-time setting, S={1, --, d} $Y = \begin{pmatrix} \mu(D) \\ \vdots \\ \mu(d) \end{pmatrix} \quad Y = VTP \quad i \land Y = VT(1-P) = 2.$ Note: the row sums of P are 1. Thus $P[i] = [i] \quad So \quad (P-T)[i] = 0$ $I \not\in \exists \forall e \not\in R^d \quad s, l'. \quad ((P - \noti)^T = Q^T)$ $w^{T}(P-I)$ We need a probability vector: $V_j \ge 0$, $\sum_{j=1}^{j} V_j = 1$. Solong as wizo, and wite we can renormalize $V = \sum_{i=1}^{l} W_{i}$



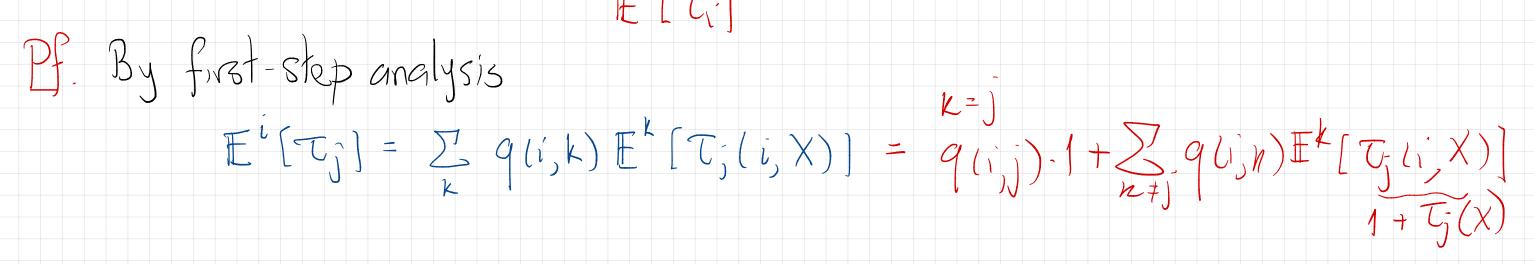


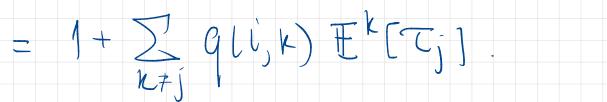


Note: the existence of an invariant distribution is not guaranteed in infinite state spaces. Eg. Simple random walk on \mathbb{Z} $q(i,j) = \frac{1}{2} I_{j=i+1} + \frac{1}{2} I_{j=i-1} = q(j,i)$ i. JA, BEIRS.t. M(j) = A+Bj. · Since M(j) >0 VjEZ, B=0. · Sme $\mu(j) = A$ must satisfy $\sum_{j \in \mathbb{Z}} \mu(j) = 1$, VIn fact, the random walk does have an invariant measure $\mu(j)=1$ $\forall j$ and it is unique up to positive multiple. But this Marka chain docs not have an invariant probability distribution.

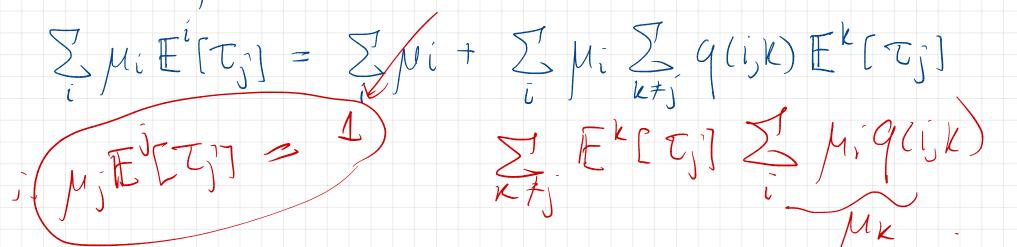
Prop: Let (Xn)nen be a finite state irreducible Markov chain. Then there is a unique invariant probability distribution μ :







. if mis an invariant distribution, then tj



We've shown that if µ is an invariant probability distribution, then

$M_i \cdot \mathbb{E}^{\nu} [\mathcal{T}_i] = 1.$

We already proved that an invariant distribution exists; .' this is it. //

Cor: In the finite state irreducible setting, the unique invariant distribution is strictly positive.

Observation: In [Lec. 44.1] we proved that $o < E'[T_i] < \infty \ \forall i$ (in the finite state irroducible case); we now have the additional fact:

$\sum_{i=1}^{n} \overline{\mathbb{E}^{i}[\mathcal{T}_{i}]} = 1$

A smilar result holds for infinite chains: every

isreducible, recurrent D'(Tica)-1 Vi chain

possesses a unique (up to scale) invariant measure, that is strictly positive.

The equation $E^{i}[\tau_{j}] = 1 + \sum_{\substack{k\neq j \\ k\neq j}} q_{i,k} E^{k}[\tau_{j}]$ says that, if $u^{(j)}$ is the vector $u^{(j)}_{i} = E^{i}[\tau_{j}]$, then $u^{(j)}_{i} = 1 + \sum_{\substack{k\neq j \\ k\neq j}} q^{(j,k)} u^{(j)}_{k}$ Note: the equation I.e. $u^{(j)} = \begin{bmatrix} i \\ i \\ i \end{bmatrix} + P^{(j)} u^{(j)} \qquad \begin{bmatrix} P^{(j)} \end{bmatrix}_{ik} = Q^{(j)} k + j$ This can be used to compute expected passage times, with linear algebra. $[E^{1}(T_{j})] = u^{(j)} = (T - P^{(j)})^{-1} [i]$. $[E^{d}(T_{j})]$ See [Driver, § 22.9.2] for many worked examples