Let $(X_t)_{t\in T}$ be a Markov process with (homogeneous) transition semigroup $(q_t)_{t\in T}$.
If the initial distribution $\mathrm{Law}(X_0)$ is μ, then for $t>0$,

$$\mathrm{Law}(X_t)(dy) = \int \mu(dx)\, q_t(x,dy) = \mu(dy) \quad \forall t\in T.$$

Def. A law μ is called invariant or stationary if

(By the Markov property: if μ is an invariant law and $\mathrm{Law}(X_{t_0}) = \mu$ for some $t_0 \in T$, then $\mathrm{Law}(\mu_t)(dy) = \int \mu(dy)\, q_{t-t_0}(x,dy) = \mu(dy)$. $\forall t \geq t_0$.)

In the discrete-time setting, we only need to check the 1-step transition:

$$\begin{aligned}\mathrm{Law}(X_{n+1})(dy) &= \int \mathrm{Law}(X_n)(dx)\, q(x,dy) \\ &\left.\begin{aligned} &= \int \mu(dx)\, q(x,dy) \\ &= \mu(dy). \end{aligned}\right\} \text{strong induction.}\end{aligned}$$

If the state space is discrete, this becomes a linear algebra question: $\mu(j) = \sum_i \mu(i)\, q(i,j)$

Let's focus on the finite state space, discrete-time setting, $S = \{1, \dots, d\}$

$$\mu(j) = \sum_i \mu(i) q(i,j) \qquad P_{ij} = q(i,j)$$

$$\underline{v} = \begin{bmatrix} \mu(1) \\ \vdots \\ \mu(d) \end{bmatrix}$$

$$\Updownarrow$$

$$\underline{v}^T = \underline{v}^T P \qquad \text{i.e. } \underline{v}^T (I - P) = \underline{0}.$$

Note: the row sums of P are 1. Thus

$$P \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{so} \quad (P - I) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \qquad \operatorname{rank}(P - I) \leq d - 1.$$

$$\operatorname{rank}(P - I) = \operatorname{rank}((P - I)^T)$$

I.e. $\exists\, \underline{w} \in \mathbb{R}^d$ s.t. $((P - I)^T \underline{w})^T = \underline{0}^T$

$$((P - I)^T \underline{w})^T = \underline{w}^T (P - I)$$

We need a probability vector: $v_j \geq 0$, $\sum_j v_j = 1$.

So long as $w_j \geq 0$, and $\underline{w} \neq \underline{0}$ we can renormalize

$$\underline{v} = \frac{1}{\sum_j w_j} \underline{w}$$

Prop: Any finite state Markov chain has an invariant distribution.

Pf. We saw on the last slide that the transition matrix P possesses a vector $\underline{w} \neq \underline{0}$ s.t. $\underline{w}^T P = \underline{w}^T$. Define $\underline{v}$ by $v_j := |w_j|$

Claim: $\underline{v}^T P = \underline{v}^T$.

$\hookrightarrow \quad v_j = |w_j| = \left|\sum_i w_i P_{ij}\right| \leq \sum_i |w_i P_{ij}| = \sum_i v_i P_{ij}$

So: if $\underline{v}^T P \neq \underline{v}^T$, it follows that there is some j for which $v_j \lneq \sum_i v_i P_{ij}$

$$\therefore \quad \sum_j v_j \lneq \sum_{i,j} v_i P_{ij} = \sum_i v_i \underbrace{\sum_j P_{ij}}_{=1} = \sum_i v_i \quad \lightning$$

Thus, indeed, $\underline{v}^T P = \underline{v}^T$. Since $\underline{v} \neq \underline{0}$ and $v_j \geq 0 \ \forall j$,

$$\mu_j := v_j \Big/ \sum_i v_i = v_j / \|v\|_1$$

defines a probability vector, and

$$\mu^T P = \left(\frac{v}{\|v\|_1}\right)^T P = \frac{1}{\|v\|_1} v^T P = \frac{1}{\|v\|_1} v^T = \mu^T. \quad ///$$

(This is a version of the Perron–Fröbenius theorem.)

Note: the existence of an invariant distribution is not guaranteed in infinite state spaces.

Eg. Simple random walk on $\mathbb{Z}$ $\quad q(i,j) = \frac{1}{2}\mathbb{1}_{j=i+1} + \frac{1}{2}\mathbb{1}_{j=i-1} = q(j,i)$

If μ satisfies $\mu(j) = \sum_i \mu(i) q(i,j)$

$$= \sum_i q(j,i)\mu(i) \Bigg\} \therefore \mu = Q\mu \quad Qf(j) = \sum_i q(j,i) f(i) \quad \forall j \in \mathbb{Z}.$$

$\therefore \exists A, B \in \mathbb{R}$ s.t. $\mu(j) = A + Bj$.

- Since $\mu(j) \geq 0 \quad \forall j \in \mathbb{Z}$, $B = 0$.
- Since $\mu(j) = A$ must satisfy $\sum_{j \in \mathbb{Z}} \mu(j) = 1$, ↯

In fact, the random walk does have an

invariant measure $\quad \mu(j) = 1 \quad \forall j$

and it is unique up to positive multiple. But this Markov chain does not have an invariant probability distribution.

Prop: Let $(X_n)_{n\in\mathbb{N}}$ be a finite state, irreducible Markov chain. Then there is a unique invariant probability distribution μ:

$$\mu_i = \frac{1}{\mathbb{E}^i[\tau_i]} > 0 .$$

Pf. By first-step analysis

$$\mathbb{E}^i[\tau_j] = \sum_k q(i,k)\,\mathbb{E}^k[\tau_j(i,X)] = \overset{k=j}{q(i,j)\cdot 1} + \sum_{k\neq j} q(i,k)\,\mathbb{E}^k[\underbrace{\tau_j(i,X)}_{1+\tau_j(X)}]$$

$$= 1 + \sum_{k\neq j} q(i,k)\,\mathbb{E}^k[\tau_j] .$$

$\therefore$ if μ is an invariant distribution, then $\forall j$

$$\sum_i \mu_i \mathbb{E}^i[\tau_j] = \sum_i \mu_i + \sum_i \mu_i \sum_{k\neq j} q(i,k)\,\mathbb{E}^k[\tau_j]$$

$$\sum_i \mu_i = 1$$

$$\sum_{k\neq j} \mathbb{E}^k[\tau_j] \underbrace{\sum_i \mu_i q(i,k)}_{\mu_k} .$$

$i=j$: $\mu_j \mathbb{E}^j[\tau_j] = 1$

We've shown that if μ is an invariant probability distribution, then

$$\mu_i \cdot \mathbb{E}^i[\tau_i] = 1.$$

We already proved that an invariant distribution exists; $\therefore$ this is it. ///

Cor: In the finite state irreducible setting, the unique invariant distribution is strictly positive.

Observation: In [Lec. 44.1] we proved that $0 < \mathbb{E}^i[\tau_i] < \infty \ \forall i$ (in the finite state irreducible case); we now have the additional fact:

$$\sum_i \frac{1}{\mathbb{E}^i[\tau_i]} = 1.$$

A similar result holds for infinite chains: every irreducible, recurrent $\mathbb{P}^i(\tau_i < \infty) = 1 \ \forall i$ chain possesses a unique (up to scale) invariant measure, that is strictly positive.

Note: the equation $\mathbb{E}^i[\tau_j] = 1 + \sum_{k \neq j} q(i,k)\,\mathbb{E}^k[\tau_j]$

says that, if $\underline{u}^{(j)}$ is the vector $u_i^{(j)} = \mathbb{E}^i[\tau_j]$, then

$$u_i^{(j)} = 1 + \sum_{k \neq j} q(i,k)\, u_k^{(j)}$$

I.e. $\underline{u}^{(j)} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + P^{(j)} \underline{u}^{(j)}$ $\qquad [P^{(j)}]_{ik} = q(i,k)\mathbb{1}_{k \neq j}$.

This can be used to compute expected passage times, with linear algebra.

$$\begin{bmatrix} \mathbb{E}^1[\tau_j] \\ \vdots \\ \mathbb{E}^d[\tau_j] \end{bmatrix} = \underline{u}^{(j)} = (I - P^{(j)})^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

See [Driver, § 22.9.2] for many worked examples.