A Condition for Finite Expectation Hitting Times

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain in $(S, \mathcal{B})$, and let $B \in \mathcal{B}$.
If q is the 1-step transition kernel, recall that $q_B = q|_{B\times \mathcal{B}|_B}$, and

$$(Q_B f)(x) = \int_B q_B(x, dy) f(y).$$

Recall that $T_B(X) = \inf\{n \geq 0 : X_n \in B\}$

We would like a general tool to guarantee that $\mathbb{E}^x[T_B] < \infty$.
The following lemma helps.

Lemma: If T is a $\mathbb{N}$-valued r.v., then for any $N \in \mathbb{N}$,

$$\mathbb{E}[T] \leq N \sum_{k=0}^{\infty} \mathbb{P}(T > Nk).$$

Pf. Just note that

$$\sum_{k=0}^{\infty} \mathbb{1}_{T > Nk}$$

Prop: Suppose there is a (uniform) $N \in \mathbb{N}$ and $\delta > 0$ s.t.

$$\mathbb{P}^x(T_B \leq N) \geq \delta \qquad \forall x \in B^c.$$

Then $\mathbb{E}^x[T_B] < \infty \quad \forall x \in S$. In fact, $\sup_{x \in S} \mathbb{E}^x[T_B] < \infty$.

Pf. For any $n \in \mathbb{N}$, $\mathbb{P}^x(T_B > n) =$

$$= \int q(x, dx_1) \int q(x_1, dx_2) \cdots \int q(x_{n-1}, dx_n)$$

But

$$\mathbb{P}^x(T_B > N) = 1 - \mathbb{P}^x(T_B \leq N) \leq 1 - \delta$$

Now, by the lemma, $\mathbb{E}^x[T_B] \leq N \sum_{k=0}^{\infty} \mathbb{P}^x(T_B > kN)$

$$= N \sum_{k=0}^{\infty}$$

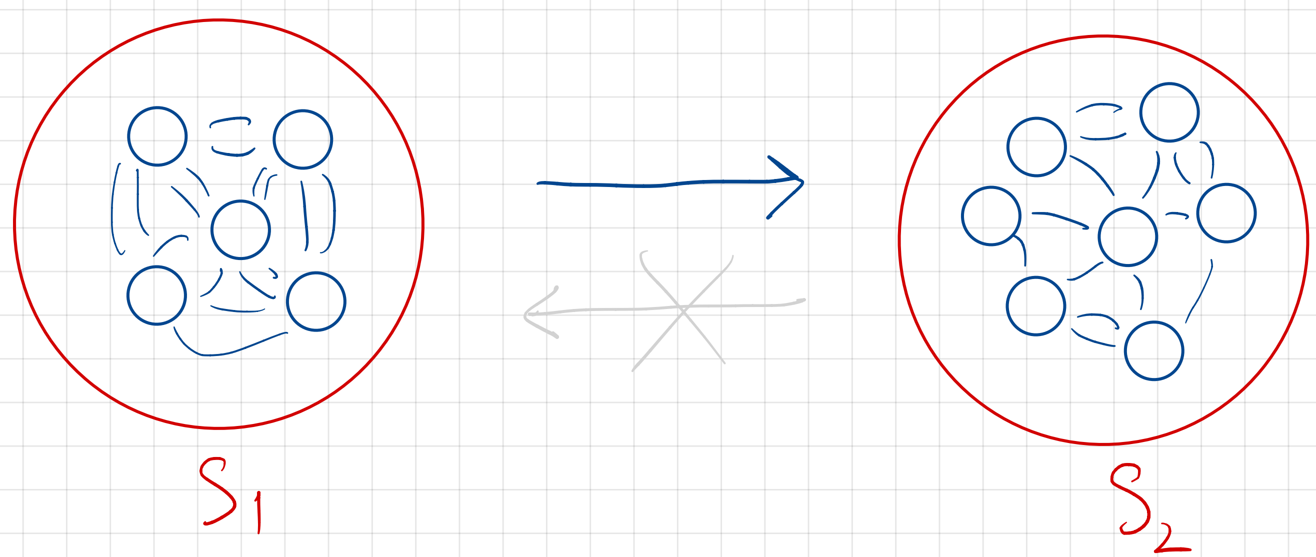
Thus, if there is a positive probability path from any point x into B

then $\mathbb{E}^x[T_B] < \infty$

One scenario where this is common is finite state chains.
But it can still fail there.

$S = S_1 \cup S_2$. For any $x \in S_2$, $y \in S_1$,
$q(x,y) = 0$.

Eg.



Here $\mathbb{P}^x[T_{S_1} = \infty]$ $\quad \forall x \in S_2$.

Def: A Markov chain with transition matrix q is irreducible if $\forall x, y \; \exists n$ s.t. $q^n(x,y) > 0$.

Cor: If $(X_n)_{n\in\mathbb{N}}$ is an irreducible Markov chain on a finite state space S,

$$\mathbb{E}^i[T_j] < \infty \quad \forall i,j\in S$$

Pf. Fix $j\in S$. By assumption, for each $i\in S$, there is some $n = n(i,j)$ s.t.

$$q^n(i,j) > 0$$

$$= \mathbb{P}^i(X_n = j)$$

$$\{X_n = j\} \subseteq \{T_j \le n\}.$$

$$\therefore \quad \mathbb{P}^i(T_j \le N) \ge \mathbb{P}^i(X_n = j)$$

The result now follows from the proposition. ///

Return Times

Suppose we start a Markov chain in state x. Will it ever return to x?

Def: Given a Markov chain $(X_n)_{n\in\mathbb{N}}$ on $(S,\mathcal{B})$, for each state $j\in S$, the passage time $\tau_j = \tau_j(X) = \inf\{n\geq 1 : X_n = j\}$.

- If $i\neq j$, $\tau_j =$ $\quad\mathbb{P}^i$-a.s.
- On the event $\{X_0 = i\}$, τ_i is the return time to i.

Prop: If $\sup_{i,j} \mathbb{E}^i[\tau_j] =: H < \infty$, then $\sup_i \mathbb{E}^i[\tau_i] \leq H+1$.

Pf. $\mathbb{E}^i[\tau_i(X)] =$

(By the Corollary, this applies to irreducible finite state chains.)