A Condition for Finite Expectation Hitting Times

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain in $(S,\mathcal{B})$, and let $B\in\mathcal{B}$.
If q is the 1-step transition kernel, recall that $q_B = q|_{B\times\mathcal{B}|_B}$, and

$$(Q_B f)(x) = \int_B q_B(x,dy)\, f(y).$$

Recall that $T_B(X) = \inf\{n\geq 0 : X_n\in B\} \in \mathbb{N}\cup\{+\infty\}$.

We would like a general tool to guarantee that $\mathbb{E}^x[T_B]<\infty$.
The following lemma helps.

Lemma: If T is a $\mathbb{N}$-valued r.v., then for any $N\in\mathbb{N}$,

$$\mathbb{E}[T] \leq N\sum_{k=0}^{\infty} \mathbb{P}(T>Nk).$$

Pf. Just note that

$$\mathbb{E}\Big(\sum_{k=0}^{\infty} \mathbb{1}_{T>Nk}\Big) = \sum_{k=0}^{\infty} \mathbb{1}_{T/N>k} = \sum_{k=0}^{\lceil (T/N)\rceil -1} 1 = \lceil (T/N)\rceil \geq \mathbb{E}\Big(\frac{T}{N}\Big).$$

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Prop: Suppose there is a (uniform) $N \in \mathbb{N}$ and $\delta > 0$ s.t.

$$\mathbb{P}^x(T_B \le N) \ge \delta \quad \forall x \in B^c.$$

Then $\mathbb{E}^x[T_B] < \infty \quad \forall x \in S$. In fact, $\sup_{x \in S} \mathbb{E}^x[T_B] < \infty$.

Pf. For any $n \in \mathbb{N}$, $\mathbb{P}^x(T_B > n) = \mathbb{P}^x(X_0 \in B^c, X_1 \in B^c, \dots, X_n \in B^c)$

$$= \mathbb{P}^x\left((X_0, X_1, \dots, X_n) \in (B^c)^n\right)$$

$\| Q_{B^c}^N 1 \|_\infty \le \alpha < 1.$

$$= \int_{B^c} q(x, dx_1) \int_{B^c} q(x_1, dx_2) \cdots \int_{B^c} q(x_{n-1}, dx_n)\, 1. = (Q_{B^c}^n 1)(x).$$

But $Q_{B^c}^N(1)(x) = \mathbb{P}^x(T_B > N) = 1 - \mathbb{P}^x(T_B \le N) \le 1 - \delta =: \alpha \in [0, 1)$.

Now, by the lemma, $\mathbb{E}^x[T_B] \le N \sum_{k=0}^{\infty} \mathbb{P}^x(T_B > kN)$

$$(Q_{B^c}^{kN} 1)(x) = (Q_{B^c}^N)^{k-1} (Q_{B^c}^N 1)(x)$$

$$\le (Q_{B^c}^N)^{k-1}(\alpha)(x)$$

$$= \alpha (Q_B^{N(k-1)} 1)(x).$$

$\therefore \| Q_{B^c}^{kN} 1 \|_\infty \le \alpha^k.$

$$= N \sum_{k=0}^{\infty} (Q_{B^c}^{kN} 1)(x)$$

$$\le N \sum_{k=0}^{\infty} \alpha^k < \infty.$$

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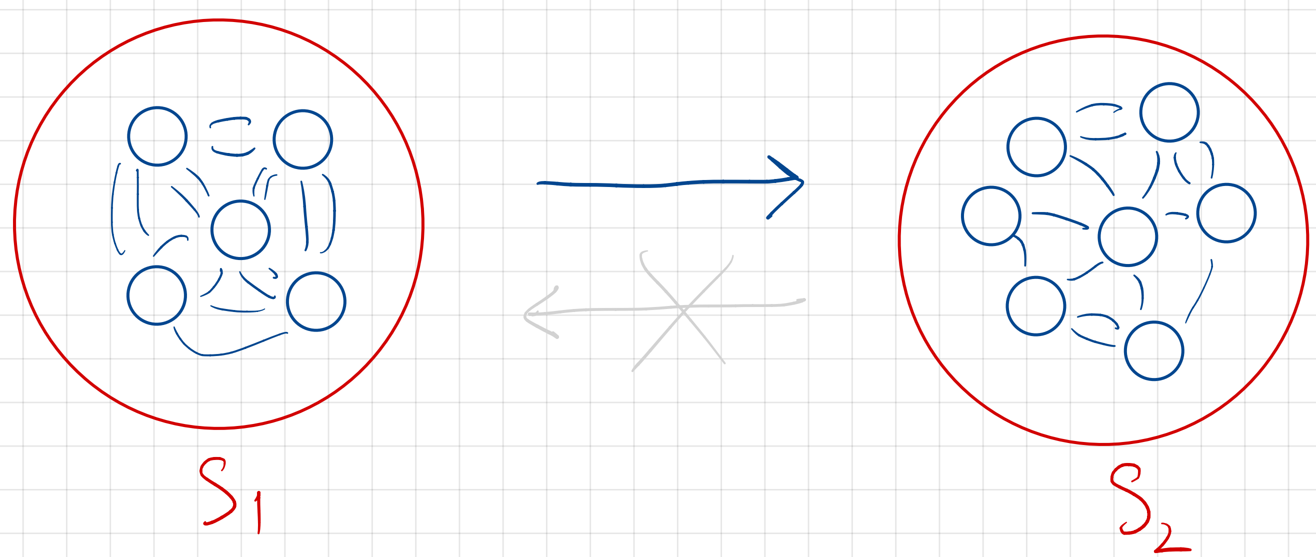
Thus, if there is a uniform over x positive probability path from any point x into B w a uniform # of steps

then $\mathbb{E}^x[T_B] < \infty$

One scenario where this is common is finite state chains.
But it can still fail there.

Eg.



$S = S_1 \sqcup S_2$. For any $x \in S_2$, $y \in S_1$,
$q(x,y) = 0$.
$q^n(x,y) = 0$.

Here $\mathbb{P}^x[T_{S_1} = \infty] = 1 \quad \forall x \in S_2$.

Def: A Markov chain with transition matrix q is **irreducible** if $\forall x,y \ \exists n$ s.t. $q^n(x,y) > 0$.

Cor: If $(X_n)_{n\in\mathbb{N}}$ is an irreducible Markov chain on a finite state space S,

$$\mathbb{E}^i[T_j] < \infty \quad \forall i,j\in S$$

$\uparrow$ $T_{\{j\}}$.

Pf. Fix $j\in S$. By assumption, for each $i\in S$, there is some $n = n(i,j)$ s.t.

Let $N = \max_{i,j} n(i,j)$

$$\delta := \min_{i,j} \delta_{ij} > 0 \qquad \delta_{ij} = q^n(i,j) > 0$$

$$\sum_{i_1,i_2,\dots,i_{n-1}\in S} q(i,i_1)\,q(i_1,i_2)\cdots q(i_{n-1},j) = \mathbb{P}^i(X_n = j)$$

$$\{X_n = j\} \subseteq \{T_j \le n\} \subseteq \{T_j \le N\}$$

$$\therefore \quad \mathbb{P}^i(T_j \le N) \ge \mathbb{P}^i(X_n = j) = \delta_{ij} \ge \delta > 0.$$

The result now follows from the proposition. ///

Return Times

Suppose we start a Markov chain in state x. Will it ever return to x?

Def: Given a Markov chain $(X_n)_{n\in\mathbb{N}}$ on $(S,\mathcal{B})$, for each state $j\in S$, the passage time $\tau_j = \tau_j(X) = \inf\{n\geq 1 : X_n = j\}$.

- If $i\neq j$, $\tau_j = T_j$ $\mathbb{P}^i$-a.s.
- On the event $\{X_0 = i\}$, τ_i is the return time to i.

Prop: If $\sup_{i,j}\mathbb{E}^i[T_j] =: H < \infty$, then $\sup_i \mathbb{E}^i[\tau_i] \leq H+1$.

Pf. $\mathbb{E}^i[\tau_i(X)] = \sum_{j\in S} q(i,j)\,\mathbb{E}^j[\tau_i(i,X)]$

$$= q(i,i)\,\mathbb{E}^i[\underbrace{\tau_i(i,X)}_{1}] + \sum_{j\neq i} q(i,j)\,\mathbb{E}^j[\underbrace{\tau_i(i,X)}_{\tau_i(X)+1}]$$

$$= 1 + \sum_{j\neq i} q(i,j)\,\mathbb{E}^j[T_i]$$

$$\leq 1 + H\sum_{j\neq i} q(i,j) \leq 1+H. \quad ///$$

(By the Corollary, this applies to irreducible finite state chains.)