(Biased) Random Walks on $\mathbb{Z}$

Fix $p \in (0,1)$. Consider the Markov chain on $\mathbb{Z}$ with transition kernel

$$q(x,y) = p\,\mathbb{1}_{y=x+1} + (1-p)\,\mathbb{1}_{y=x-1}.$$

(Simple random walk is the case $p = \frac{1}{2}$.)

Let $B \subseteq \mathbb{Z}$. We can use first-step analysis to understand T_B and $L_B = X_{T_B}$.

$$u(x) = \mathbb{E}^x[h(X_{T_B}) : T_B < \infty]$$

We know $u = h$ on B, and $u = Qu$ on B^c

$$u(x) = \sum_y q(x,y)\,u(y) = p\,u(x+1) + (1-p)\,u(x-1)$$

3-term recursion $p\,u(x+1) - u(x) + (1-p)\,u(x-1) \qquad x \notin B$

General solution: for $x \notin B$,

$$\left\{\begin{array}{lll} p \neq \frac{1}{2} & u(x) = \alpha + \beta \lambda_p^x & \lambda_p = \frac{1-p}{p} \\ p = \frac{1}{2} & u(x) = \alpha + \beta x & \end{array}\right\} \text{ for some } \alpha, \beta \in \mathbb{C}$$

Symmetry: $q(-x, -y) = p \, \mathbb{1}_{\{-y = -x+1\}} + (1-p) \, \mathbb{1}_{\{-y = -x-1\}}$

I.e. for the associated Markov chain $(X_n)_{n \in \mathbb{N}}$, $(-X_n)_{n \in \mathbb{N}}$ is the "same" chain with $p \leftrightarrow 1-p$.

WLOG take $p \in [\frac{1}{2}, 1)$.

We'd like to determine α, β.

We'll consider the case

$$B = \{a, b\} \quad \text{with } a < b.$$

As we'll see, there's a nice relationship between $T_{\{a,b\}}$ and T_a, T_b.

$T_{\{a,b\}} = \min\{n \geq 0 : X_n \in \{a,b\}\}$

We will compute $u(x) = \mathbb{P}^x(T_b < T_a) = \mathbb{P}^x(L_{\{a,b\}} = b)$
for $a < x < b$

By first-step analysis,

$$u(x) = p\,u(x+1) + (1-p)\,u(x-1), \quad a < x < b$$

$$\therefore \quad u(x) = \alpha + \beta \lambda_p^x \text{ where } \lambda_p = \frac{1-p}{p}, \text{ for some } \alpha, \beta.$$

Also: $u(a) = \mathbb{P}^a(T_b < T_a)$
$u(b) = \mathbb{P}^b(T_b < T_a)$

$$\therefore \quad \alpha + \beta \lambda_p^a = 0$$
$$\alpha + \beta \lambda_p^b = 1$$

$$\therefore \quad u(x) = \frac{\lambda_p^x - \lambda_p^a}{\lambda_p^b - \lambda_p^a} = \frac{1 - \lambda_p^{x-a}}{1 - \lambda_p^{b-a}}$$

Gambler's Ruin

You play a game against the House; the probability of the House winning is p

X_n = House's fortune (your losses) after n plays.

Your initial fortune is $\$x$ ($x<0$ as we're viewing from the House perspective).

You play repeatedly (winning or losing $\$1$ each play) until you go broke
or you win $\$b$

$$\mathbb{P}(\text{You win } \$b, \text{ never going broke}) = \mathbb{P}^x(T_b < T_0)$$

$$= \frac{1-\lambda_p^x}{1-\lambda_p^b} = \frac{1-(1/\lambda_p)^{|x|}}{1-(1/\lambda_p)^{|b|}}$$

Eg. $|b| = 2|x|$ (want to double your fortune)

What about just T_b?

$\mathbb{P}^x(T_b < \infty)$? $\mathbb{E}^x[T_b]$?

As $a \downarrow -\infty$, $\mathbb{P}^x(T_a < \infty) \downarrow 0$ for any fixed x,

$\therefore$ If $x \leq b$, $\mathbb{P}^x(T_b < \infty) = \lim\limits_{a \downarrow -\infty} \mathbb{P}^x(T_b < T_a) = \lim\limits_{a \downarrow -\infty} \dfrac{1 - \lambda_p^{x-a}}{1 - \lambda_p^{b-a}}$

On the other hand, $\{T_a > T_b\} \downarrow \{T_a = \infty\}$ as $b \uparrow \infty$, so

if $x \geq a$, $\mathbb{P}^x(T_a = \infty) = \lim\limits_{b \uparrow \infty} \mathbb{P}^x(T_b < T_a) = \lim\limits_{b \uparrow \infty} \dfrac{1 - \lambda_p^{x-a}}{1 - \lambda_p^{b-a}}$

Putting these together yields:

Cor: If $\mathbb{P}(X_{n+1} = x+1 \mid X_n = x) = p > \frac{1}{2}$, then for any $x, b \in \mathbb{Z}$,

$$\mathbb{P}^x(T_b < \infty) = \begin{cases} \\ \\ \end{cases}$$

In particular: in the case $b < x$,

$$\mathbb{P}^x(T_b = \infty)$$

$$\therefore \mathbb{E}^x[T_b]$$

What about $b > x$? Here $\mathbb{P}^x(T_b < \infty) = 1$. But is $\mathbb{E}^x[T_b] < \infty$?

A similar 1st step analysis shows that

$$\mathbb{E}^x[T_b] = \begin{cases} \infty, & b < x \\ \end{cases}$$

See [Driver, Example 22.51] for details.