(Biased) Random Walks on $\mathbb{Z}$

Fix $p \in (0,1)$. Consider the Markov chain on $\mathbb{Z}$ with transition kernel

$$q(x,y) = p\,\mathbb{1}_{y=x+1} + (1-p)\,\mathbb{1}_{y=x-1}.$$

(Simple random walk is the case $p = \frac{1}{2}$.)

We can construct this process explicitly: $X_n = \underbrace{\xi_0}_{=x} + \underbrace{\xi_1 + \xi_2 + \xi_3 + \cdots + \xi_n}_{\text{iid.} \overset{d}{=} p\delta_1 + (1-p)\delta_{-1}}$

Let $B \subseteq \mathbb{Z}$. We can use first-step analysis to understand T_B and $L_B = X_{T_B}$.

$$u(x) = \mathbb{E}^x[h(X_{T_B}) : T_B < \infty] \qquad h \equiv 1,\ u(x) = \mathbb{P}^x(T_B < \infty).$$

We know $u = h$ on B, and $u = Qu$ on B^c

$$u(x) = \sum_y q(x,y)\,u(y) = p\,u(x+1) + (1-p)\,u(x-1)$$

3-term recursion $p\,u(x+1) - u(x) + (1-p)\,u(x-1) = 0 \quad x \notin B$

char. polynomial: $p\lambda^2 - \lambda + (1-p) = 0$.

roots: $\lambda = 1, \quad \lambda = \lambda_p = \frac{1-p}{p}$.

General solution: for $x \notin B$,

$$\left\{\begin{array}{lll} p \neq \frac{1}{2} & u(x) = \alpha + \beta \lambda_p^x & \lambda_p = \frac{1-p}{p} \\ p = \frac{1}{2} & u(x) = \alpha + \beta x & \end{array}\right\} \text{ for some } \alpha, \beta \in \mathbb{C}$$

Symmetry: $q(-x,-y) = p \mathbb{1}_{\{-y=-x+1\}} + (1-p)\mathbb{1}_{\{-y=-x-1\}}$

$$= p \mathbb{1}_{\{y=x-1\}} + (1-p)\mathbb{1}_{\{y=x+1\}}$$

I.e. for the associated Markov chain $(X_n)_{n\in\mathbb{N}}$, $(-X_n)_{n\in\mathbb{N}}$ is the "same" chain with $p \leftrightarrow 1-p$.

WLOG take $p \in [\frac{1}{2}, 1)$.

already studied $p = \frac{1}{2}$, $B = \{b\}$.

We'd like to determine α, β. functions of B, h.

We'll consider the case

$$B = \{a, b\} \quad \text{with} \quad a < b.$$

As we'll see, there's a nice relationship between $T_{\{a,b\}}$ and T_a, T_b.

$$T_{\{a,b\}} = \min\{n \geq 0 : X_n \in \{a,b\}\}$$

We will compute $u(x) = \mathbb{P}^x(T_b < T_a) = \mathbb{P}^x(L_{\{a,b\}} = b)$

for $a < x < b$

$= \mathbb{E}^x[h(X_{T_{\{a,b\}}}) : T_{\{a,b\}} < \infty]$

where $h(b) = 1$, $h(a) = 0$.

By first-step analysis,

$$u(x) = p\,u(x+1) + (1-p)\,u(x-1), \quad a < x < b$$

$\therefore$ $u(x) = \alpha + \beta \lambda_p^x$ where $\lambda_p = \frac{1-p}{p}$, for some α, β. $(p > \frac{1}{2})$

Also: $u(a) = \mathbb{P}^a(T_b < T_a) = 0$

$u(b) = \mathbb{P}^b(T_b < T_a) = 1$

$\therefore$ $\left.\begin{aligned} \alpha + \beta\lambda_p^a &= 0 \\ \alpha + \beta\lambda_p^b &= 1 \end{aligned}\right\}$ $\quad \begin{aligned} \alpha &= -\lambda_p^a(\lambda_p^b - \lambda_p^a)^{-1} \\ \beta &= (\lambda_p^b - \lambda_p^a)^{-1} \end{aligned}$

$$\therefore\ u(x) = \frac{\lambda_p^x - \lambda_p^a}{\lambda_p^b - \lambda_p^a} = \frac{1 - \lambda_p^{x-a}}{1 - \lambda_p^{b-a}} = \frac{x-a}{b-a}.$$

($p = \frac{1}{2}$, take $\lim_{p \downarrow \frac{1}{2}}$)

Gambler's Ruin

You play a game against the House; the probability of the House winning is $p > \frac{1}{2}$.

X_n = House's fortune (your losses) after n plays.

Your initial fortune is $\$x$ ($x<0$ as we're viewing from the House perspective).

You play repeatedly (winning or losing \$1 each play) until you go broke $X_n = 0$

or you win $\$b$ $X_n = b < x < 0$.

$$\mathbb{P}(\text{You win } \$b, \text{ never going broke}) = \mathbb{P}^x(T_b < T_0)$$

$$= \frac{1-\lambda_p^x}{1-\lambda_p^b} = \frac{1-(1/\lambda_p)^{|x|}}{1-(1/\lambda_p)^{|b|}} \qquad |x|<|b| \qquad \lambda_p = \frac{1-p}{p} < 1.$$

Eg. $|b| = 2|x|$ (want to double your fortune)

$$= \frac{1-(1/\lambda_p)^{|x|}}{1-(1/\lambda_p)^{2|x|}}$$

$|x| = 100$

$\cancel{0.30}$ $\frac{6}{10^6}$

$$= \frac{1}{1+(1/\lambda_p)^{|x|}} < \lambda_p^{|x|}$$

Eg. $p = 0.53$

$\therefore \lambda_p = 0.89$

What about just T_b?

$\mathbb{P}^x(T_b < \infty)$? $\mathbb{E}^x[T_b]$?

As $a \downarrow -\infty$, $\mathbb{P}^x(T_a < \infty) \downarrow 0$ for any fixed x, $\lambda_p = \frac{1-p}{p} < 1$

$\therefore$ If $x \leq b$, $\mathbb{P}^x(T_b < \infty) = \lim_{a \downarrow -\infty} \mathbb{P}^x(T_b < T_a) = \lim_{a \downarrow -\infty} \frac{1-\lambda_p^{x-a}}{1-\lambda_p^{b-a}} = 1$.

On the other hand, $\{T_a > T_b\} \downarrow \{T_a = \infty\}$ as $b \uparrow \infty$, so

if $x \geq a$, $\mathbb{P}^x(T_a = \infty) = \lim_{b \uparrow \infty} \mathbb{P}^x(T_b < T_a) = \lim_{b \uparrow \infty} \frac{1-\lambda_p^{x-a}}{1-\lambda_p^{b-a}} = 1 - \lambda_p^{x-a}$.

Putting these together yields:

Cor: If $\mathbb{P}(X_{n+1} = x+1 \mid X_n = x) = p > \frac{1}{2}$, then for any $x, b \in \mathbb{Z}$,

$$\mathbb{P}^x(T_b < \infty) = \begin{cases} 1 & \text{if } x \leq b \\ \left(\frac{1-p}{p}\right)^{x-b} & \text{if } x \geq b. \end{cases}$$

In particular: in the case $b < x$,

$$\mathbb{P}^x(T_b = \infty) > 0.$$

$$\therefore \mathbb{E}^x[T_b] = \infty.$$

What about $b > x$? Here $\mathbb{P}^x(T_b < \infty) = 1$. But is $\mathbb{E}^x[T_b] < \infty$?

A similar 1st step analysis shows that

$$\mathbb{E}^x[T_b] = \begin{cases} \infty, & b < x \\ \dfrac{b-x}{p-(1-p)} & b \geq x \end{cases}$$

See [Driver, Example 22.51] for details.