Markov chain: $(X_n)_{n\in\mathbb{N}} : (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P}) \to (S, \mathcal{B})$

Hitting time $\quad T_B(X) = \min\{n \geq 0 : X_n \in B\}$

We'd like to understand the distribution of T_B: e.g. $\mathbb{P}^x(T_B < \infty)$, $\mathbb{E}^x[T_B]$.

we can study more general quantities; think in terms of games.

Eg. $\mathbb{E}^x[h(X_{T_B}) : \quad]$

↳ start at x, continue until you hit B. At the point y where you enter B, collect \$ $h(y)$.

Eg. $\mathbb{E}^x\left[\sum_{n=0}^{T_B} g(X_n)\right]$

↳ start at x, follow random path $x_1, x_2, \ldots, x_N$ until $x_N \in B$. Pick up \$ $g(x_k)$ each time you visit x_k. How much do you collect total?

Sub-Probability Kernels.

Let $B \in \mathcal{B}$. Recall that $\mathcal{B}|_B := \{E \in \mathcal{B} : E \subseteq B\}$

a σ-field over B.

If $q : S \times \mathcal{B} \to [0,1]$ is a probability kernel, we can restrict it to B:

$$q_B : B \times \mathcal{B}|_B \to [0,1].$$

$$q_B(x, E) =$$

Note: $\int q_B(x, dy) =$

As usual, we identify q_B with an operator

$$Q_B : \mathbb{B}(S, \mathcal{B}) \to \mathbb{B}(S, \mathcal{B})$$
$$(Q_B f)(x) = \int q_B(x, dy) f(y)$$

Has all the same regularity properties as Q,

except $Q_B(1)$

Prop: Let $h \in \mathbb{B}(S, \mathcal{B})$ or $h : S \to [0, \infty]$ $\mathcal{B}/\mathcal{B}[0,\infty]$-measurable.
Let $(X_n)_{n \in \mathbb{N}}$ be a Markov process in $(S, \mathcal{B})$ with 1-step transition operator Q. Let $B \in \mathcal{B}$. Define

$$u(x) := \mathbb{E}^x[\, h(X_{T_B}) : T_B < \infty \,], \quad x \in S.$$

Then $u \in \mathbb{B}(S, \mathcal{B})$ or $u : S \to [0, \infty]$ is $\mathcal{B}/\mathcal{B}[0,\infty]$-measurable, and $u = Qu$ on B^c, $u = h$ on B.

Moreover, $u = Q_B h + Q_{B^c} u$ on B^c.

Pf. We use first step analysis:

$$\mathbb{E}^x[F(X_0,X_1,X_2,\dots)] = \int_S q(x,dy)\,\mathbb{E}^y[F(x,X_0,X_1,X_2,\dots)].$$

Here $F(X_0,X_1,X_2,\dots) = h(X_{T_B})\mathbb{1}_{T_B<\infty}$.

Note, if $x \in B^c$, then

and

$\therefore\ F(x,X_0,X_1,X_2,\dots) =$

So $u(x) = \mathbb{E}^x[F(X_0,X_1,X_2,\dots)]$

$$= \int_S q(x,dy)\,\mathbb{E}^y[F(x,X_0,X_1,X_2,\dots)]$$

Prop: Let $g : B^c \to [0,\infty]$ be $\mathcal{B}/\mathcal{B}[0,\infty])$-measurable. Set

$$V(x) := \mathbb{E}^x\Big[\sum_{0 \le n < T_B} g(X_n)\Big], \qquad x \in S$$

Then $V = 0$ on B, $V = QV + g =$ on B^c.

Pf. Set $F(X_0, X_1, X_2, \dots) = \sum_{0 \le n < T_B} g(X_n)$. Take $x \in B^c$.

$\therefore F(x, X_0, X_1, X_2, \dots) =$

$\therefore V(x) = \mathbb{E}^x[F(X_0, X_1, X_2, \dots)] = \int_S q(x, dy)\, \mathbb{E}^y[F(x, X_0, X_1, X_2, \dots)]$

So, we have to solve recursive equations

$$u = Q_{B^c} u + Q_B h \quad \text{on } B^c,$$

$$v = Q_{B^c} v + g \quad \text{on } B^c$$

If $I - Q_B$ is invertible, and we can compute its inverse, we're good.

This is true if $\mathbb{P}^x(T_B < \infty) = 1 \ \forall x \in B^c$ See [Driver, Cor 22.58]

but otherwise is generally <u>false</u>.

In that case, we need to characterize which solution is the right one.

Theorem: In the problems considered in the last two propositions, u and v are the unique minimal solutions of the recursion equations.

See [Driver, § 22.8]