Markov chain: $(X_n)_{n\in\mathbb{N}} : (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P}) \to (S, \mathcal{B})$

Hitting time $T_B(X) = \min\{n \geq 0 : X_n \in B\}$ $B \in \mathcal{B}$.

$\mathbb{N} \cup \{+\infty\}$ $:= \infty$ if $X_n \notin B \ \forall n$.

we'd like to understand the distribution of T_B: e.g. $\mathbb{P}^x(T_B < \infty)$, $\mathbb{E}^x[T_B]$.

we can study more general quantities;

think in terms of games.

E.g. $\mathbb{E}^x[h(X_{T_B}) : T_B < \infty]$ $\leftarrow$ E.g. $h \equiv 1$, $\mathbb{E}^x[\mathbb{1}_{T_B<\infty}] = \mathbb{P}^x(T_B < \infty)$

↳ start at x, continue until you hit B.

At the point y where you enter B,

collect \$ $h(y)$.

E.g. $\mathbb{E}^x\left[\sum_{n=0}^{T_B} g(X_n)\right]$

↳ start at x, follow random path $x_1, x_2, \ldots, x_N$

until $x_N \in B$. Pick up \$ $g(x_k)$ each time

you visit x_k. How much do you collect total?

Sub-Probability Kernels.

Let $B \in \mathcal{B}$. Recall that $\mathcal{B}|_B := \{E \in \mathcal{B} : E \subseteq B\} \overset{\text{b/c } \mathcal{B} \text{ is a field.}}{=} \{F \cap B : F \in \mathcal{B}\}$

a σ-field over B.

If $q : S \times \mathcal{B} \to [0,1]$ is a probability kernel, we can restrict it to B:

$$q_B : B \times \mathcal{B}|_B \to [0,1].$$

$$q_B(x, E) = q(x, E) \quad \forall x \in B,\ E \in \mathcal{B}|_B.$$

$q = q_B + q_{B^c}$

$Q = Q_B + Q_{B^c}$

Note: $\int q_B(x, dy) = q(x, B) \neq 1$ nec. $(q(x,S) = 1)$

≤ 1.

As usual, we identify q_B with an operator

$$Q_B : \mathbb{B}(S, \mathcal{B}) \to \mathbb{B}(S, \mathcal{B}) \quad (\text{or } Q_B : \{\geq 0 \text{ meas. fns}\}$$

$$(Q_B f)(x) = \int_B q_B(x, dy) f(y) = Q(f \mathbb{1}_B)(x).$$

Has all the same regularity properties as Q,

except $Q_B(1) = Q(\mathbb{1}_B) \neq 1$
≤ 1

Prop: Let $h \in \mathbb{B}(S,\mathcal{B})$ or $h : S \to [0,\infty]$ $\mathcal{B}/\mathcal{B}[0,\infty]$-measurable.
Let $(X_n)_{n\in\mathbb{N}}$ be a Markov process in $(S,\mathcal{B})$ with 1-step transition operator Q. Let $B \in \mathcal{B}$. Define

$$u(x) := \mathbb{E}^x[\,h(X_{T_B}) : T_B < \infty], \quad x \in S.$$

Then $u \in \mathbb{B}(S,\mathcal{B})$ or $u : S \to [0,\infty]$ is $\mathcal{B}/\mathcal{B}[0,\infty]$-measurable,
and $u = Qu$ on B^c, $u = h$ on B. $\leftarrow$ If $x \in B$, $\underset{(<\infty)}{T_B = 0}$ $\therefore h(X_{T_B}) = h(X_0) = h(x)$.

Moreover, $u = Q_B h + Q_{B^c} u$ on B^c.

I.e. $u(x) = \int_B q(x,dy)h(y) + \int_{B^c} q(x,dy)u(y) \quad \forall x \in B^c$.

In particular: if $h \equiv 1$, $u(x) = \mathbb{P}^x(T_B < \infty)$,

$$u = Q_{B^c}u + \underset{Q(\mathbb{1}_B)}{Q_B(1)} \quad \text{on } B^c.$$

$u \equiv 1$ on B.

Pf. We use first step analysis:

$$\mathbb{E}^x[F(X_0,X_1,X_2,\dots)] = \int_S q(x,dy)\,\mathbb{E}^y[F(x,X_0,X_1,X_2,\dots)].$$

Here $F(X_0,X_1,X_2,\dots) = h(X_{T_B})\mathbb{1}_{T_B<\infty}$

$$= h(L_B)\mathbb{1}_{T_B<\infty}.$$

Note, if $x\in B^c$, then $L_B(x,X_0,X_1,X_2,\dots) = L_B(X_0,X_1,X_2,\dots)$

and $T_B(x,X_0,X_1,X_2,\dots)<\infty$ iff $T_B(X_0,X_1,X_2,\dots)<\infty$.

$$\therefore\ F(x,X_0,X_1,X_2,\dots) = F(X_0,X_1,X_2,\dots)$$

So
$$\begin{aligned}
u(x) &= \mathbb{E}^x[F(X_0,X_1,X_2,\dots)]\\
&= \int_S q(x,dy)\,\mathbb{E}^y[F(x,X_0,X_1,X_2,\dots)]\\
&= \int_S q(x,dy)\,\mathbb{E}^y[F(X_0,X_1,X_2,\dots)] = \int_S q(x,dy)u(y)\\
&= Qu(x) = (Q_{B^c}u)(x) + (Q_B u)(x)
\end{aligned}$$

$$\underbrace{\int_B q(x,dy)\,u(y)}_{(Q_B h)(x)}.\ ///$$

Prop: Let $g : B^c \to [0,\infty]$ be $\mathcal{B}/\mathcal{B}([0,\infty])$-measurable. Set

$$v(x) := \mathbb{E}^x\Big[\sum_{0\le n<T_B} g(X_n)\Big], \quad x \in S \qquad \text{Eg. } g = \mathbb{1}_{B^c},\ v(x) = \mathbb{E}^x[T_B].$$

Then $v = 0$ on B, $\quad v = Qv + g = Q_{B^c}v + g$ on B^c.

If $g = \mathbb{1}_{B^c}$,

$$\forall x \in B^c, \quad \mathbb{E}^x[T_B] = v(x) = Q_{B^c}v(x) + 1 = \int_{B^c} q(x,dy)\,\mathbb{E}^y[T_B] + 1.$$

Pf. Set $F(X_0, X_1, X_2, \dots) = \sum_{0\le n<T_B} g(X_n)$. Take $x \in B^c$.

$$\therefore\ F(x, X_0, X_1, X_2, \dots) = \begin{cases} g(x), & \text{if } X_0 \in B \\ g(x) + F(X_0, X_1, X_2, \dots) & \text{if } X_0 \notin B. \end{cases}$$

$$= g(x) + \mathbb{1}_{X_0\in B^c} F(X_0, X_1, X_2, \dots).$$

$$\therefore\ v(x) = \mathbb{E}^x[F(X_0, X_1, X_2, \dots)] = \int_S q(x,dy)\,\mathbb{E}^y[F(x, X_0, X_1, X_2, \dots)]$$

$$= g(x) + \int_{B^c} q(x,dy)\,\mathbb{E}^y[\cancel{\mathbb{1}_{X_0\in B^c}} F(X_0, X_1, X_2, \dots)]$$

$$= g(x) + Q_{B^c}v(x) = g(x) + Q(v\cancel{\mathbb{1}_{B^c}})(x) \qquad ///$$

So, we have to solve recursive equations

$u = Q_{B^c} u + Q_B h$ on B^c, $u = h$ on B $\therefore$ u "=" $(I - Q_{B^c})^{-1} Q_B h$

$v = Q_{B^c} v + g$ on B^c $v = 0$ on B. v "=" $(I - Q_{B^c})^{-1} g$ on B^c.

If $I - Q_B$ is invertible, and we can compute its inverse, we're good.

This is true if $\mathbb{P}^x(T_B < \infty) = 1$ $\forall x \in B^c$ See [Driver, Cor 22.58]
but otherwise is generally false.

In that case, we need to characterize which solution is the right one.

Theorem: In the problems considered in the last two propositions, u and v are the unique minimal solutions of the recursion equations.
See [Driver, §22.8]