Hitting Times for Discrete-Time Markov Chains

$(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ filtered probability space $(S, \mathcal{B})$ measurable space

$X = X_n : (\Omega, \mathcal{F}_n) \to (S, \mathcal{B})$ adapted time-homogeneous Markov Chain

For any $B \in \mathcal{B}$ $T_B = T_B(X) :=$

first hitting time of B by X.

More generally, for any path $\underline{x} = (x_0, x_1, x_2, \dots) \in S^{\mathbb{N}}$, we define

$$T_B(\underline{x}) :=$$

Then $T_B(X)(\omega) = T_B(X_0(\omega), X_1(\omega), X_2(\omega), \dots)$.

We also have the hitting location $L_B(\underline{x}) :=$

Basic Observations:

1. If $x_0 \in B^c$, and $T_B(\underline{x}) < \infty$, then $L_B(x_0, x_1, x_2, \dots)$

2. If $x_0 \in B^c$, then $T_B(\underline{x}) =$

First Step

Let q be the 1-step transition kernel for $(X_n)_{n\in\mathbb{N}}$.

Theorem: Let $F \in \mathbb{B}(S^{\mathbb{N}}, \mathcal{B}^{\otimes \mathbb{N}})$.

(Also allowed: $F : S^{\mathbb{N}} \to [0,\infty]$ $\mathcal{B}^{\otimes\mathbb{N}}/\mathcal{B}([0,\infty])$ - measurable.)

Then for any $x \in S$,

$$\mathbb{E}^x[F(\underset{(X_0,\, X_1,\, X_2,\, \dots}{X})] = \int_S q(x,dy)\, \mathbb{E}^y[F(\underset{(x,\, X_0,\, X_1,\, X_2,\, \dots)}{x, X})]$$

Pf. Since $X_0 = x$ $\mathbb{P}^x$-a.s., $F(X_0, X_1, \dots) = F(x, X_1, X_2, \dots)$ $\mathbb{P}^x$-a.s.

By the Markov property: [Lec. 38.2]

$$\mathbb{E}^x[F(X_0, X_1, \dots) \mid \mathcal{F}_1] = \mathbb{E}^x[F(x, X_1, X_2, \dots) \mid \mathcal{F}_1]$$

Now take expectations:

$$\mathbb{E}^x[F(X)] =$$

Alternate Direct Proof (for $F(x_0, x_1, x_2, \ldots) = f(x_0, x_1, \ldots, x_n)$):

$$\mathbb{E}^x[f(X_0, X_1, \ldots, X_n)] = \int_{S^n} q(x, dx_1)\, q(x_1, dx_2) \cdots q(x_{n-1}, dx_n)\, f(x, x_1, x_2, \ldots, x_n)$$

Note: by induction, for any finite n,

$$\mathbb{E}^x[F(X_0, X_1, \ldots)] = \int_{S^n} \prod_{j=1}^{n} q(x_{j-1}, dx_j)\, \mathbb{E}^{x_n}[F(x_0, x_1, \ldots, x_{n-1}, X_0, X_1, \ldots)].$$

In the discrete space setting, the statement is:

$$\mathbb{E}^x[F(X_0, X_1, \ldots)] = \sum_{y \in S} q(x, y)\, \mathbb{E}^y[F(x, X_0, X_1, \ldots)]$$

Armed with our simple observations, this can be an effective tool for analyzing hitting times.

Eg. Simple random walk on $\mathbb{Z}$.

$$q(x, x\pm 1) = \frac{1}{2}, \quad q(x,y) = 0 \text{ if } |y-x| \neq 1.$$

Let $\mathbb{B} = \{b\}$, $b \in \mathbb{Z}$. What can we say about T_b?

- Is $T_b < \infty$? I.e. $\mathbb{P}^x(T_b < \infty) > 0$?
- Distribution of T_b? $\mathbb{E}^x[T_b]$?

Idea: let $u(x) = \mathbb{E}^x[T_b]$ Use first step analysis.

$$u(x) = \mathbb{E}^x[T_b(X)]$$

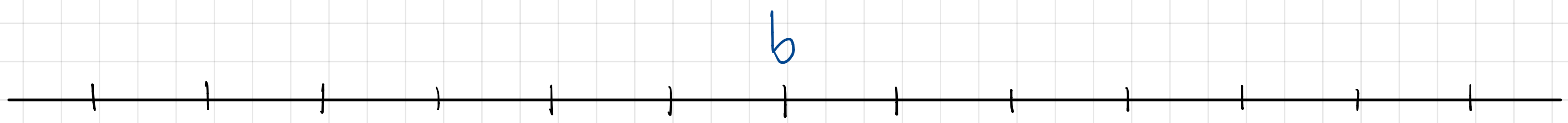
$E^x[T_b] = u(x) = \frac{1}{2}(u(x-1) + u(x+1)) + 1$. $x \neq b$

Note: $\mathbb{P}^b(T_b = 0) = 1$, $\therefore$ $\mathbb{E}^b[T_b] = 0$ $=$

. One possible solution: $u(x) =$ Are there other solutions?

Claim: If $u(x) = \infty$ for some $x < b$, then $u(y) = \infty$ for all $y < b$.

" " $x > b$ " " $y > b$.



So, we could have $T_b = \infty$ on one side,

$T_b < \infty$ on the other side.

or $T_b < \infty$ everywhere.

Suppose $u(x) < \infty$ for $x > b$. Here's a clever trick:

$$w(x) := u(x) + x^2 .$$

$\therefore$ For $x > b$, $\frac{1}{2}\left(w(x-1) + w(x+1)\right)$

On [HW] you will show that the general finite solution is

$$w(x) = A_0 + A_1 x , \quad x > b \text{ for some } A_0, A_1 \in \mathbb{R} .$$

But then $u(x) = A_0 + A_1 x - x^2$

The same argument for $x < b$ shows that:

$$\mathbb{E}^x[T_b] = u(x)$$

However: $\mathbb{P}^x(T_b < \infty)$