Hitting Times for Discrete-Time Markov Chains

$(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ filtered probability space $\quad (S, \mathcal{B})$ measurable space

$X = X_n : (\Omega, \mathcal{F}_n) \to (S, \mathcal{B})$ adapted time-homogeneous Markov Chain

For any $B \in \mathcal{B}$ $\quad T_B = T_B(X) := \min\{n \geq 0 : X_n \in B\}$. $\quad (\min \emptyset = \infty)$.

$\mathbb{N}$-valued r.v. $\quad$ first hitting time of B by X.

More generally, for any path $\quad \underline{x} = (x_0, x_1, x_2, \dots) \in S^{\mathbb{N}}$, we define

$$T_B(\underline{x}) := \min\{n \geq 0 : x_n \in B\}$$

Then $T_B(X)(\omega) = T_B(X_0(\omega), X_1(\omega), X_2(\omega), \dots)$.

We also have the hitting location $\quad L_B(\underline{x}) := x_{T_B(\underline{x})}$

Basic Observations:

1. If $x_0 \in B^c$, and $T_B(\underline{x}) < \infty$, then $L_B(x_0, x_1, x_2, \dots)$
$$= L_B(x_1, x_2, x_3, \dots)$$

2. If $x_0 \in B^c$, then $T_B(\underline{x}) = 1 + \mathbb{1}_{B^c}(x_1)\, T_B(x_1, x_2, \dots)$

First Step

Let q be the 1-step transition kernel for $(X_n)_{n\in\mathbb{N}}$.

Theorem: Let $F \in \mathbb{B}(S^{\mathbb{N}}, \mathcal{B}^{\otimes\mathbb{N}})$.

(Also allowed: $F: S^{\mathbb{N}} \to [0,\infty]$ $\mathcal{B}^{\otimes\mathbb{N}}/\mathcal{B}([0,\infty])$ - measurable.)

Then for any $x \in S$, E.g. $F = \mathbb{1}_B$

$$\mathbb{E}^x[F(\underbrace{X}_{(X_0, X_1, X_2, \dots)})] = \int_S q(x,dy)\, \mathbb{E}^y[F(\underbrace{x, X}_{(x, X_0, X_1, X_2, \dots)})]$$

Pf. Since $X_0 = x$ $\mathbb{P}^x$-a.s., $F(X_0, X_1, \dots) = F(x, X_1, X_2, \dots)$ $\mathbb{P}^x$-a.s.

By the Markov property: [Lec. 38.2]

$$\mathbb{E}^x\Big(\mathbb{E}^x[F(X_0, X_1, \dots) \mid \mathcal{F}_1]\Big) = \mathbb{E}^x[F(x, X_1, X_2, \dots) \mid \mathcal{F}_1]$$
$$= \mathbb{E}^x\Big(\mathbb{E}^{X_1}[F(x, X_0, X_1, \dots)]\Big)$$

Now take expectations:

$$\mathbb{E}^x[F(X)] = \mathbb{E}^x[\mathbb{E}^{X_1}[F(x, X_0, X_1, \dots)]]$$
$$= \int_S q(x,dy)\, \mathbb{E}^y[F(x, X)]. \quad ///$$

Alternate Direct Proof (for $F(x_0, x_1, x_2, \dots) = f(x_0, x_1, \dots, x_n)$):

$$\mathbb{E}^x[f(X_0, X_1, \dots, X_n)] = \int_{S^n} q(x, dx_1)\, q(x_1, d\underset{x_1}{\cancel{x_2}}) \cdots q(x_{n\cancel{-1}}, d\underset{x_{n-1}}{\cancel{x_n}})\, f(x, x_1, x_2, \dots, x_n)$$

$$= \int_S q(x, dy) \underbrace{\int_{S^{n-1}} q(y, dx_1) \cdots q(x_{n-2}, dx_{n-1})\, f(x, y, x_1, \dots, x_{n-1})}$$

$$= \int_S q(x, dy)\, \mathbb{E}^y[f(x, \underset{y}{\cancel{x_0}}, x_1, \dots, x_{n-1})] \qquad ///$$

Note: by induction, for any finite n,

$$\mathbb{E}^x[F(X_0, X_1, \dots)] = \int_{S^n} \prod_{j=1}^{n} q(x_{j-1}, dx_j)\, \mathbb{E}^{x_n}[F(x_0, x_1, \dots, x_{n-1}, X_0, X_1, \dots)].$$

In the discrete space setting, the statement is:

$$\mathbb{E}^x[F(X_0, X_1, \dots)] = \sum_{y \in S} q(x, y)\, \mathbb{E}^y[F(x, X_0, X_1, \dots)]$$

Armed with our simple observations, this can be an effective tool for analyzing hitting times.

Eg. Simple random walk on $\mathbb{Z}$.

$$q(x, x\pm 1) = \frac{1}{2}, \quad q(x,y) = 0 \text{ if } |y-x| \neq 1.$$

Let $B = \{b\}$, $b \in \mathbb{Z}$. What can we say about T_b?

- Is $T_b < \infty$? I.e. $\mathbb{P}^x(T_b < \infty) > 0$? "recurrence"
- Distribution of T_b? $\mathbb{E}^x[T_b]$?

Idea: let $u(x) = \mathbb{E}^x[T_b] \in [0, \infty]$ Use first step analysis.

$$u(x) = \mathbb{E}^x[T_b(X)]$$

$$= \sum_{y \in \mathbb{Z}} q(x,y)\, \mathbb{E}^y[T_b(x, X)]$$

$$= \frac{1}{2}\, \mathbb{E}^{x-1}[\underbrace{T_b(x, X)}_{[1+T_b(X)]}] + \frac{1}{2}\, \mathbb{E}^{x+1}[\underbrace{T_b(x, X)}_{[1+T_b(X)]}]$$

$$= 1 + \frac{1}{2} u(x-1) + \frac{1}{2} u(x+1).$$

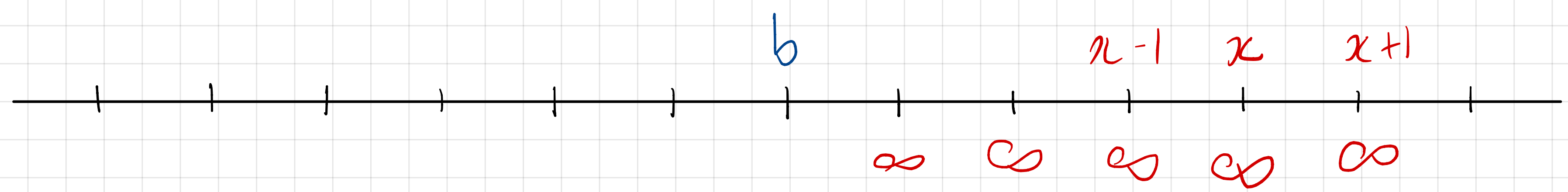
$$E^x[T_b] = u(x) = \tfrac{1}{2}\big(u(x-1) + u(x+1)\big) + 1. \qquad x \neq b$$

Note: $\mathbb{P}^b(T_b = 0) = 1$, $\therefore\ \mathbb{E}^b[T_b] = 0 = u(b)$

. One possible solution: $u(x) = \begin{cases} 0 & x = b \\ \infty & x \neq b. \end{cases}$ Are there other solutions?

Claim: If $u(x) = \infty$ for some $x < b$, then $u(y) = \infty$ for all $y < b$.

" " $x > b$ " " $y > b$.



$$u(x+1) = \tfrac{1}{2}\big(u(x) + u(x+2)\big) + 1 = \infty$$

$$u(x-1) = \tfrac{1}{2}\big(u(x-2) + u(x)\big) + 1 = \infty$$

So, we could have $T_b = \infty$ on one side,

$T_b < \infty$ on the other side.

or $T_b < \infty$ everywhere.

Suppose $u(x) < \infty$ for $x > b$. Here's a clever trick:

$$w(x) := u(x) + x^2.$$

$\therefore$ For $x > b$, $\frac{1}{2}(w(x-1) + w(x+1)) = \frac{1}{2}\left(u(x-1) + (x-1)^2 + u(x+1) + (x+1)^2 \right)$

$$= \frac{1}{2}\left(u(x-1) + u(x+1)\right) + x^2 + 1$$

$$= u(x) - \cancel{1} + x^2 + \cancel{1} = w(x).$$

On [HW] you will show that the general finite solution is

$$w(x) = A_0 + A_1 x, \quad x > b \text{ for some } A_0, A_1 \in \mathbb{R}.$$

But then $u(x) = A_0 + A_1 x - x^2 \downarrow -\infty$ as $x \uparrow \infty$.

$\lightning$ $u(x) \geq 0 \ \forall x > b$.

The same argument for $x < b$ shows that:

$$\mathbb{E}^x[T_b] = u(x) = \infty \ \forall x \neq b. \quad \text{"null recurrent"}.$$

However: $\mathbb{P}^x(T_b < \infty) = \underline{1} \ \forall x$. [HW].