In a discrete (homogeneous) time, countable state space Markov chain, a state i is called absorbing if $q(i,i) = 1$.

In general, if there's any loop, $q(i,i) > 0$, the Markov chain is called lazy. Assuming no absorbing states, we can define a kernel

$$\tilde{q}(i,j) = \begin{cases} \end{cases}$$

We can describe the Markov chain with kernel $\tilde{q}$.

Prop: (Jump-Hold) Let $(Y_n)_{n \in \mathbb{N}}$ be a time homogeneous Markov chain with 1-step transition kernel q.

Set $\tau_1 = \inf\{n > 0 : Y_n \neq Y_0\}$, $\tau_{k+1} = \inf\{n > \tau_k : Y_n \neq Y_{\tau_k}\}$.

Then $(Z_k := Y_{\tau_k})_{k \in \mathbb{N}}$ is a Markov chain with 1-step transition kernel $\tilde{q}$. Moreover, wrt. $\mathbb{P}(\cdot \mid Z_1 = i_1, \dots, Z_k = i_k)$, $\{\sigma_k := \tau_k - \tau_{k-1}\}_{k=1}^{n}$ are independent

Pf. [HW]

We now develop a similar description of <u>any</u> (operator norm continuous) homogeneous continuous time Markov chain.

Theorem: Let $(X_t)_{t\geq 0}$ be a Markov chain with bounded generator A (so A has matrix a satisfying $a_i := -a(i,i) = \sum_{j\neq i} a(i,j)$, $\sup_i a_i < \infty$).
Assume no absorbing states: $a_i > 0 \ \forall i$.
Let $J_0 = 0$, $J_1 = \inf\{t \geq 0 : X_t \neq X_0\}$, and generally $J_k = \inf\{t > J_{k-1} : X_t \neq X_{J_{k-1}}\}$.
Set $S_k = J_k - J_{k-1}$.

1. $Z_k := X_{J_k}$ is a Markov chain with 1-step transition kernel mass function

$$\tilde{p}(i,j) = \begin{cases} \dfrac{a(i,j)}{a_i}, & i \neq j \\ 0, & i = j \end{cases}$$

2. If $i_0, i_1, \dots, i_n$ are states with $i_0 \neq i_1, i_1 \neq i_2, \dots, i_{n-1} \neq i_n$, then wrt $\mathbb{P}(\,\cdot \mid Z_0 = i_0, Z_1 = i_1, \dots, Z_n = i_n)$, $S_1, S_2, \dots, S_n$ are independent with $S_k \overset{d}{=} \operatorname{Exp}(a_{i_{k-1}})$.

Pf. We previously showed that, if $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain with 1-step transition operator $P = I + \frac{1}{\lambda} A$ (where $\lambda \geq \frac{1}{2}\|A\|_{op} = \sup_i a_i$), then X_t can be realized as Y_{N_t}.

$$p = I + \frac{1}{\lambda} a$$

i.e. $$p(i,j) = \delta_{ij} + \frac{1}{\lambda} a(i,j)$$

Let $\{\tau_k\}_{k\in\mathbb{N}}$ be the jump times of the "lazy" chain $(Y_n)_{n\in\mathbb{N}}$. By the discrete Jump-Hold proposition, $Z_k := Y_{\tau_k}$ is a Markov chain with transition kernel $\tilde{p}(i,i) = 0$, and for $i \neq j$,

$$\tilde{p}(i,j) = \frac{p(i,j)}{1 - p(i,i)}$$

Now, $X_t = Y_{N_t}$ and so $X_{J_k} = Y_{\tau_k} = Z_k$.

This proves part 1.

For 2, let $\sigma_k = \tau_k - \tau_{k-1}$. Fix states $i_0 \neq i_1, i_1 \neq i_2, \dots, i_{n-1} \neq i_n$.

Let $B = \{ Z_0 = i_0, Z_1 = i_1, \dots, Z_n = i_n \}$.

By the discrete Jump-Hold proposition, relative to $\mathbb{P}(\cdot \mid B)$,

$\sigma_1, \dots, \sigma_n$ are independent, with $\sigma_k \overset{d}{=} \mathrm{Geom}(1 - p(i_{k-1}, i_{k-1}))$

Note that $\tau_k = \sigma_1 + \dots + \sigma_k$.

Similarly: $J_k = (k^{th}$ jump time of $X_t) = T_1 + T_2 + \dots + T_{\tau_k}$

where T_ℓ are the jump times of N_t. They are $\overset{d}{=} \mathrm{Exp}(\lambda)$, and independent.

Lemma: Let $T_\ell \overset{d}{=} \mathrm{Exp}(\lambda)$ for $\ell \in \mathbb{N}$,

$\{\sigma_1, \dots, \sigma_n\}$ satisfy $\sigma_k \overset{d}{=} \mathrm{Geom}(b_k)$,

and $\{T_\ell\}_{\ell \in \mathbb{N}} \cup \{\sigma_1, \dots, \sigma_n\}$ are independent.

Set $W_n = T_1 + \dots + T_n$ & $S_\ell = W_{\tau_\ell} - W_{\tau_{\ell-1}}$.

$\tau_k = \sigma_1 + \dots + \sigma_k$

Then $\{S_1, \dots, S_n\}$ are independent, with

$$S_\ell \overset{d}{=} \mathrm{Exp}(b_\ell \cdot \lambda)$$

[HW].

This Jump-Hold description applies to all continuous homogeneous time Markov chains that are operator norm continuous, i.e. $\inf_i q_t(i,i) \to 1$ as $t \downarrow 0$, that have no absorbing states.

However, in many interesting examples, $\|Q_t - I\|_{op} \not\to 0$ as $t \downarrow 0$.

This is equivalent to having a bounded generator $Q_t = e^{tA}$

It is perfectly possible for Q_t to be a Markov semigroup (of bounded operators) of the form $Q_t = e^{tA}$, where A is not bounded.

We will explore this more in later lectures.