Under the continuity condition $\|Q_t - I\|_{op} \to 0$, $(Q_t)_{t\geq 0}$ has a bounded generator A.
In the (countable) discrete setting, Q_t has matrix q_t, and we showed that

$$Af(i) = \sum_j a(i,j) f(j), \quad \text{where} \quad a(i,j) = \frac{d}{dt} q_t(i,j)\Big|_{t=0}.$$

It's the generator, so $Q_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = I + tA + O(t^2)$.

$$\therefore \quad q_t(i,j) = \delta_{ij} + t\, a(i,j) + O(t^2).$$

Now, if $(X_t)_{t\geq 0}$ is a Markov chain with transition kernels $(q_t)_{t\geq 0}$, we have $\mathbb{P}(X_t = j \mid X_0 = i) = q_t(i,j) = \delta_{ij} + t\, a(i,j) + O(t^2)$.

The $a(i,j)$ are the transition rates of the process.
I.e. if $i \neq j$, then $\mathbb{P}(X_t = j \mid X_0 = i) \cong a(i,j)t$ for small $t > 0$.
More precisely $\frac{d}{dt} \mathbb{P}(X_t = j \mid X_0 = i)\Big|_{t=0^+} = a(i,j)$

For this reason, this is called the bounded rates case.

$\inf_i q_t(i,i) \to 1$ as $t \downarrow 0$.

$$\infty > \sup_i \sum_j |a(i,j)| = \sup_i \Big(|a(i,i)| + \sum_{j \neq i} \cancel{|} a(i,j) \cancel{|} \Big) = 2 \sup_i |a(i,i)|$$

The generator matrix a completely describes the evolution of the process.
But how?

$e^{t \cdot \frac{1}{\lambda} A} = Q_{t/\lambda}$ – time-rescaled.

First: rescaling a produces a new generator: $\frac{1}{\lambda} a$ for some $\lambda > 0$

Let $p = I + \frac{1}{\lambda} a$ $\quad \|p\|_\infty \leq 1 + \frac{1}{\lambda} \|a\|_\infty < \infty$.

If $\lambda > \frac{1}{2}\|a\|_\infty = \sup_i |a(i,i)|$, $p(i,i) = 1 + \frac{1}{\lambda} a(i,i) \geq 0$ $\quad \left|\frac{a(i,i)}{\lambda}\right| \leq 1$.

and $\sum_j p(i,j) = \sum_j \left(\delta_{ij} + \frac{1}{\lambda} a(i,j)\right) = 1 + \frac{1}{\lambda} \cdot 0 = 1 \ \forall i$.

Thus, p is a Markov matrix: it is the 1-step transition operator for some discrete time Markov chain $(Y_n)_{n \in \mathbb{N}}$ on the same state space.

Theorem: Let $(Y_n)_{n\in\mathbb{N}}$ be a Markov chain with 1-step transition matrix $p = I + \frac{1}{\lambda} a$. Let $(N_t)_{t \geq 0}$ be a Poisson process with intensity $\lambda \geq \frac{1}{2}\|a\|_\infty$, independent from $(Y_n)_{n\in\mathbb{N}}$. Then $X_t = Y_{N_t}$ is a Markov process with generator A whose matrix is a.

Theorem: Let A be a bounded generator on $B(S)$ (for a discrete countable state space S), with matrix a. Fix some $\lambda \geq \frac{1}{2}\|a\|_\infty$. Let $(Y_n)_{n\in\mathbb{N}}$ be a Markov chain with 1-step transition operator $P = I + \frac{1}{\lambda}A$. Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity λ, independent from $(Y_n)_{n\in\mathbb{N}}$.

Then $X_t = Y_{N_t}$ is a Markov process with generator $A = \lambda(P-I)$.

Pf. $e^{tA} = e^{\lambda t(P-I)} = e^{-\lambda t} e^{\lambda t P} = e^{-\lambda t}\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} P^n = \sum_{n=0}^{\infty} \mathbb{P}(N_{t+s} - N_s = n) P^n$

Now $\mathbb{P}(N_t = n) = \mathbb{P}(N_{t+s} - N_s = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$.

We can now compute all f.d. distributions of $(X_t)_{t\geq 0}$:

For $0 = t_0 < t_1 < \dots < t_k < \infty$, and $i_0, i_1, \dots, i_k \in S$,

$$\mathbb{P}(X_0 = i_0, X_{t_1} = i_1, \dots, X_{t_k} = i_k)$$

$$= \sum_{0 \leq n_1 \leq \dots \leq n_k} \underbrace{\mathbb{P}^{i_0}(Y_{n_1} = i_1, \dots, Y_{n_k} = i_k)}_{\text{use Markov prop. to express in terms of } P} \cdot \underbrace{\mathbb{P}(N_{t_1} = n_1, \dots, N_{t_k} = n_k)}_{\text{write in terms of increments}}$$

$$\mathbb{P}(N_{t_1}=n_1,\dots,N_{t_k}=n_k) = \mathbb{P}(N_{t_1}=n_1,\dots,N_{t_{k-1}}=n_{k-1}, N_{t_k}-N_{t_{k-1}}=n_k-n_{k-1})$$

$$= \mathbb{P}(N_{t_1}=n_1,\dots,N_{t_{k-1}}=n_{k-1})\,\mathbb{P}(N_{t_k}-N_{t_{k-1}}=n_k-n_{k-1})$$

$$\mathbb{P}^{i_0}(Y_{n_1}=i_1,\dots,Y_{n_k}=i_k) = \underbrace{\mathbb{P}(Y_0=i_0)\,q^Y_{0,n_1}(i_0,i_1)\,q^Y_{n_1,n_2}(i_1,i_2)\cdots q^Y_{n_{k-2},n_{k-1}}(i_{k-2},i_{k-1})}_{\mathbb{P}^{i_0}(Y_{n_1}=i_1,\dots,Y_{n_{k-1}}=i_{k-1})}\,\underbrace{q^Y_{n_{k-1},n_k}(i_{k-1},i_k)}_{P^{n_k-n_{k-1}}(i_{k-1},i_k)}$$

Thus $\mathbb{P}(X_{t_0}=i_0, X_{t_1}=i_1,\dots,X_{t_k}=i_k)$

$$= \sum_{0\le n_1\le\cdots\le n_k} \mathbb{P}^{i_0}(Y_{n_1}=i_1,\dots,Y_{n_k}=i_k)\cdot\mathbb{P}(N_{t_1}=n_1,\dots,N_{t_k}=n_k)$$

$$= \sum_{0\le n_1\le\cdots\le n_{k-1}} \mathbb{P}^{i_0}(Y_{n_1}=i_1,\dots,Y_{n_{k-1}}=i_{k-1})\cdot\mathbb{P}(N_{t_1}=n_1,\dots,N_{t_{k-1}}=n_{k-1})$$

$$\cdot\underbrace{\sum_{m=0}^{\infty}\mathbb{P}(N_{t_k}-N_{t_{k-1}}=m)\,P^m(i_{k-1},i_k)}_{e^{(t_k-t_{k-1})A}(i_{k-1},i_k)}.$$

By induction, $$= \prod_{\ell=1}^{k} \underbrace{e^{(t_\ell-t_{\ell-1})A}(i_{\ell-1},i_\ell)}_{q_{t_{\ell-1},t_\ell}(i_{\ell-1},i_\ell)}$$

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Note: we showed explicitly that the f.d. distributions of $X_t = Y_{N_t}$ match those of a Markov process with transition semigroup e^{tA}.

Thus $(X_t)_{t\geq 0}$ is a Markov process wrt $(\mathcal{F}_t^X)_{t\geq 0}$.

If we want to be a little more explicit about the filtration, we could state the result as follows.

Theorem: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space, and let $(Y_n)_{n\in\mathbb{N}}$ be an adapted time-homogeneous Markov chain (countable state space) with 1-step transition operator P. Let $(N_t)_{t\geq 0}$ be a Poisson process with intens. λ independent from $\mathcal{F}_\infty = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right)$. Define a new filtration $\mathcal{G}_t \subseteq \mathcal{F}$ by

$$B \in \mathcal{G}_t \quad \text{iff} \quad \forall n \in \mathbb{N} \quad B \cap \{N_t = n\} \in \sigma(\mathcal{F}_t^N \cup \mathcal{F}_n)$$

Then $(\mathcal{G}_t)_{t\geq 0}$ is a filtration, $X_t = Y_{N_t}$ is adapted to it, and X_t is a time homogeneous Markov process (rel. to $\mathcal{G}_t$) with transition operators $Q_t = e^{\lambda t(P-I)}$.

[Driver, Thm 22.34]