We've now seen that if $(Q_t)_{t\geq 0}$ is operator norm continuous @ $t=0$, then there is a generator A – a bounded operator on $\mathbb{B}(S,\mathcal{B})$ s.t.

$$Q_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n .$$

One case where this level of regularity is commonplace is for discrete state spaces: S countable, $\mathcal{B} = 2^S$. Here

$$Q_t f(i) = \sum_{j\in S} q_t(i,j) f(j) \qquad \forall i \in S$$

where $(q_t)_{t\geq 0}$ are the transition kernel mass functions

Note: $q_t(i,j) \in [0,1]$, $\sum_{j\in S} q_t(i,j)$

$\therefore$ As a matrix, $\| q_t \|_\infty = \sup_{i\in S} \sum_{j\in S} |q_t(i,j)|$

The continuity condition becomes

$$\| Q_t - I \|_{op} = \| q_t - I \|_\infty = \sup_i \sum_j | q_t(i,j) - \delta_{ij} |$$

Thus, if $(q_t)_{t\geq 0}$ are Markov transition kernel mass functions on a discrete state space S, satisfying

$$\lim_{t\downarrow 0} \inf_{i\in S} q_t(i,i) = 1$$

then the transition semigroup $(Q_t)_{t\geq 0}$ has a bounded generator A.

Question: What can we say about A?

$$Af(i) = \frac{d}{dt}\Big|_{t=0^+} Q_t f(i)$$

This suggests that A has a matrix a given by

$$a(i,j) = \frac{d}{dt} q_t(i,j)\Big|_{t=0^+}$$

If S is infinite, this takes some work to prove.

First question: does A even have a matrix?

You might think this must always be true: that every (bounded) linear operator

$$T: \mathbb{B}(S) \to \mathbb{B}(S)$$

has a matrix. Following the finite-dimensional case, we would expand

$$f =$$

If f is a simple function, so $f(j)=0$ for all but finitely many $j \in S$,

$$Tf =$$

So we would expect that T has matrix $\theta(i,j) =$

But if f is not simple, there's no way to extend this:

even if $\|\theta\|_\infty < \infty$, we can't check if $Tf(i) = \sum_j \theta(i,j) f(j)$.

Basic Problem:

Fact: $\exists$ bounded operators on $\mathbb{B}(S)$ that have no matrix.

Prop: Let $\{T_n\}_{n\in\mathbb{N}}$ be bounded operators on $\mathbb{B}(S)$, each given by a matrix θ_n:

$$T_n f(i) = \sum_{j\in S} \theta_n(i,j) f(j).$$

If T is a bounded operator and $\|T_n - T\|_{op} \to 0$, then T has a matrix θ given by $\theta(i,j) =$

Pf. First, $T_n(\mathbb{1}_{\{j\}})(i) = \sum_{k\in S} \theta_n(i,k)\mathbb{1}_{\{j\}}(k) = \theta_n(i,j)$

$$\therefore \lim_{n\to\infty} \theta_n(i,j) = \lim_{n\to\infty} T_n(\mathbb{1}_{\{j\}})(i)$$

$$|T_n(\mathbb{1}_{\{j\}})(i) - T(\mathbb{1}_{\{j\}})(i)|$$

Now, $\{T_n\}_{n\in\mathbb{N}}$ is convergent, hence Cauchy.

$$(T_n - T_m) f(i) = T_n f(i) - T_m f(i)$$

$\therefore$ $T_n - T_m$ has matrix $\theta_n - \theta_m$,

$$\|\theta_n - \theta_m\|_\infty = \|T_n - T_m\|_{op}$$

$$\| \quad \sup_i \sum_j |\theta_n(i,j) - \theta_m(i,j)|$$

$\therefore$ For each i, $\sum_j |\theta_n(i,j) - \theta(i,j)| = \sum_j \lim_{m\to\infty} |\theta_n(i,j) - \theta_m(i,j)|$

As this is true for all i, taking $\sup_i$

$\therefore$ By the triangle inequality, $\|\theta\|_\infty < \infty$, so it defines a bounded operator

$$\hat{T}f(i) := \sum_j \theta(i,j) f(j).$$

But then $\|\hat{T} - T_n\|_{op} = \|\theta - \theta_n\|_\infty$

Since $\|T - T_n\|_{op} \to 0$, it follows that

Cor: Under the continuity condition $\|Q_t - I\|_{op} \to 0$ as $t \downarrow 0$, the semigroup has generator A with matrix a

$$a(i,j) = \frac{d}{dt}\Big|_{t=0^+} q_t(i,j).$$

Pf. We proved last lecture that $\left\| A - \frac{Q_t - I}{t} \right\|_{op} \to 0$ as $t \downarrow 0$.

Take any $t_n \downarrow 0$,

$\therefore$ By our proposition, A has matrix

$$a(i,j) = \lim_{n\to\infty}$$

Cor: $a(i,j) \geq 0$ for $i \neq j$, and $\sum_{j \in S} a(i,j) = 0 \quad \forall i \in S$.

(Exactly the same as the proof in [Lec.39.2].)