We've now seen that if $(Q_t)_{t\geq 0}$ is operator norm continuous @ $t=0$, then there is a generator A – a bounded operator on $\mathbb{B}(S,\mathcal{B})$ s.t.

$$Q_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n .$$

One case where this level of regularity is commonplace is for discrete state spaces: S countable, $\mathcal{B} = 2^S$. Here

$$Q_t f(i) = \sum_{j\in S} q_t(i,j) f(j) \qquad \forall i \in S$$

where $(q_t)_{t\geq 0}$ are the transition kernel mass functions

Note: $q_t(i,j) \in [0,1]$, $\sum_{j\in S} q_t(i,j) = 1 \quad \forall i \in S$.

$\therefore$ As a matrix, $\| q_t \|_\infty = \sup_{i\in S} \sum_{j\in S} \cancel{|} q_t(i,j) \cancel{|} = \sup_i 1 = 1$.

The continuity condition becomes

$$\| Q_t - I \|_{op} = \| q_t - I \|_\infty = \sup_i \sum_j | q_t(i,j) - \delta_{ij} |$$

$$= \sup_i \Big(1 - q_t(i,i) + \sum_{j\neq i} q_t(i,j) \Big) = \sup_i 2(1 - q_t(i,i)) \to 0 \text{ as } t \downarrow 0.$$

Thus, if $(q_t)_{t\geq 0}$ are Markov transition kernel mass functions on a discrete state space S, satisfying

$$\lim_{t\downarrow 1} \inf_{i\in S} q_t(i,i) = 1$$

then the transition semigroup $(Q_t)_{t\geq 0}$ has a bounded generator A.

Question: What can we say about A? $Q_t = e^{tA}$

$$Af(i) = \frac{d}{dt}\Big|_{t=0^+} Q_t f(i)$$

$$= \frac{d}{dt}\Big|_{t=0^+} \sum_{j\in S} q_t(i,j) f(j)$$

$$\overset{?}{=} \sum_{j\in S} \frac{d}{dt}\Big|_{t=0^+} q_t(i,j) f(j)$$

This suggests that A has a matrix a given by

$$a(i,j) = \frac{d}{dt} q_t(i,j)\Big|_{t=0^+}$$

If S is infinite, this takes some work to prove.

First question: does A even have a matrix?

You might think this must always be true: that every (bounded) linear operator

$$T : \mathbb{B}(S) \to \mathbb{B}(S)$$

has a matrix. Following the finite-dimensional case, we would expand

$$f = \sum_{j \in S} f(j)\, \mathbb{1}_{\{j\}}.$$

If f is a simple function, so $f(j) = 0$ for all but finitely many $j \in S$,

$$Tf(i) = \sum_{j \in S} f(j)\, T(\mathbb{1}_{\{j\}})(i)$$

So we would expect that T has matrix $\theta(i,j) = T(\mathbb{1}_{\{j\}})(i)$.

But if f is not simple, there's no way to extend this:
even if $\|\theta\|_\infty < \infty$, we can't check if $Tf(i) = \sum_j \theta(i,j) f(j)$.

Basic Problem: $f \equiv 1$ on $\mathbb{N}$. $\quad \|f - f_k\|_\infty = 1$.

$f_k = \mathbb{1}_{\{1,2,\dots,k\}}$

Fact: $\exists$ bounded operators on $\mathbb{B}(S)$ that have no matrix.

Prop: Let $\{T_n\}_{n\in\mathbb{N}}$ be bounded operators on $\mathbb{B}(S)$, each given by a matrix Θ_n:

$$T_n f(i) = \sum_{j\in S} \Theta_n(i,j) f(j).$$

$\|\Theta_n\|_\infty = \|T_n\|_{op} < \infty \ \forall n.$

If T is a bounded operator and $\|T_n - T\|_{op} \to 0$, then T has a matrix Θ given by $\Theta(i,j) = \lim_{n\to\infty} \Theta_n(i,j)$.

Pf. First, $T_n(\mathbb{1}_{\{j\}})(i) = \sum_{k\in S} \Theta_n(i,k) \mathbb{1}_{\{j\}}(k) = \Theta_n(i,j)$

$$\therefore \lim_{n\to\infty} \Theta_n(i,j) = \lim_{n\to\infty} T_n(\mathbb{1}_{\{j\}})(i) = T(\mathbb{1}_{\{j\}})(i).$$

$$|T_n(\mathbb{1}_{\{j\}})(i) - T(\mathbb{1}_{\{j\}})(i)| \leq \sup_i |\ \text{''}\ \ - \ \text{''}\ | = \|T_n(\mathbb{1}_{\{j\}}) - T(\mathbb{1}_{\{j\}})\|_\infty$$

$$\leq \|T_n - T\|_{op} \|\mathbb{1}_{\{j\}}\|_\infty \to 0.$$

Now, $\{T_n\}_{n\in\mathbb{N}}$ is convergent, hence Cauchy.

$$(T_n - T_m) f(i) = T_n f(i) - T_m f(i)$$

$$= \sum_j \Theta_n(i,j) f(j) - \sum_j \Theta_m(i,j) f(j)$$

$$= \sum_j [\Theta_n(i,j) - \Theta_m(i,j)] f(j)$$

$\therefore$ $T_n - T_m$ has matrix $\theta_n - \theta_m$, so

$\| \theta_n - \theta_m \|_\infty = \| T_n - T_m \|_{op} \to 0$ as $n, m \to \infty$.

$$\overset{\|}{\sup_i \sum_j |\theta_n(i,j) - \theta_m(i,j)|}$$

$\therefore$ For each i, $\sum_j |\theta_n(i,j) - \theta(i,j)| = \sum_j \liminf_{m\to\infty} |\theta_n(i,j) - \theta_m(i,j)|$

$\overset{\text{Fatou}}{\leq} \liminf_{m\to\infty} \sum_j |\theta_n(i,j) - \theta_m(i,j)| \leq \liminf_{m\to\infty} \|T_n - T_m\|_{op}$.

$= \|T_n - T\|_{op}$.

As this is true for all i, taking $\sup_i$ $\therefore$ $\|\theta_n - \theta\|_\infty \leq \|T_n - T\|_{op}$

$\therefore$ By the triangle inequality, $\|\theta\|_\infty < \infty$, so it defines a bounded operator

$$\hat{T} f(i) := \sum_j \theta(i,j) f(j).$$

But then $\|\hat{T} - T_n\|_{op} = \|\theta - \theta_n\|_\infty \leq \|T - T_n\|_{op} \to 0$

Since $\|T - T_n\|_{op} \to 0$, it follows that $T = \hat{T}$.

I.e. $Tf(i) = \sum_j \theta(i,j) f(j)$.

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Cor: Under the continuity condition $\|Q_t - I\|_{op} \to 0$ i.e. $\inf_i q_t(i,i) \to 1$ as $t \downarrow 0$, the semigroup has generator A with matrix a

$$a(i,j) = \frac{d}{dt}\Big|_{t=0^+} q_t(i,j).$$

Pf. We proved last lecture that $\|A - \frac{Q_t - I}{t}\|_{op} \to 0$ as $t \downarrow 0$.

Take any $t_n \downarrow 0$, $\therefore$ $\|A - \underbrace{\frac{Q_{t_n} - I}{t_n}}_{\text{has matrix } \frac{1}{t_n}[q_{t_n}(i,j) - \delta_{ij}]}\|_{op} \to 0$

$\therefore$ By our proposition, A has matrix

$$a(i,j) = \lim_{n\to\infty} \frac{1}{t_n}[q_{t_n}(i,j) - \delta_{ij}]$$
$$= \frac{d}{dt}\Big|_{t=0^+} q_t(i,j).$$

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Cor: $a(i,j) \geq 0$ for $i \neq j$, and $\sum_{j\in S} a(i,j) = 0$ $\forall i \in S$.

(Exactly the same as the proof in [Lec. 39.2].)