Theorem: Let $(Q_t)_{t\geq 0}$ be Markov transition operators over $(S, \mathcal{B})$.
Suppose $t \mapsto Q_t$ is operator norm continuous @ $t=0$:

$$\lim_{t \downarrow 0} \|Q_t - I\|_{op} = 0.$$

Then $t \mapsto Q_t$ is operator norm differentiable on $[0,\infty)$.
Let $A := \frac{d}{dt} Q_t \big|_{t=0^+}$. Then $\|A\|_{op} < \infty$, and

$$Q_t = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \quad \text{(which converges unif. in op. norm).}$$

In particular, Q_t satisfies the Kolmogorov forward and backward ODEs:

$$\frac{d}{dt} Q_t = Q_t A = A Q_t, \quad Q_0 = I.$$

Eg. For a Poisson process, $q_t(i,j) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \mathbb{1}_{j\geq i}$.

$$\therefore \|q_t - I\|_\infty = \sup_i \sum_j |q_t(i,j) - \delta_{ij}|$$

Pf. First, note that $\|Q_t\|_{op}$ $\forall t \geq 0$.

Let $t \geq 0$. If $h > 0$,

$$\|Q_{t+h} - Q_t\|_{op} = \|Q_h Q_t - Q_t\|_{op}$$

Similarly, if $t > h$,

$$\|Q_{t-h} - Q_t\|_{op} =$$

This shows $t \mapsto Q_t$ is operator norm continuous on $[0,\infty)$.

Similarly: $$\frac{Q_{t+h} - Q_t}{h} =$$

Thus, for any $t \geq 0$, and any bounded op. B,

$$\left\| \frac{Q_{t+h} - Q_t}{h} - Q_t B \right\|_{op}$$

This shows $\left(\frac{d}{dt}\right)_+ Q_t$ exists at any pt. t iff $A = \frac{d}{dt} Q_t \Big|_{t=0^+}$ exists, in which case

Thus, to show $A = \frac{d}{dt} Q_t |_{t=0^+}$ exists

it suffices to show that $t \mapsto Q_t$ is (right) diff'ble at some $t \geq 0$.

To prove this, we employ a trick due to Lars Gårding.

Gårding's trick: For $\varepsilon > 0$, define an operator B_ε on $\mathbb{B}(S, \mathcal{B})$ by

$$B_\varepsilon = \frac{1}{\varepsilon} \int_0^\varepsilon Q_s \, ds$$

Note: $B_\varepsilon Q_t f = \frac{1}{\varepsilon} \int_0^\varepsilon Q_s (Q_t f) \, ds$

I.e. $B_\varepsilon Q_t = \frac{1}{\varepsilon} \int_0^\varepsilon Q_{s+t} \, ds$

It follows by the Fundamental Theorem of Calculus that $t \mapsto B_\varepsilon Q_t$ is differentiable, and

$$\frac{d}{dt} B_\varepsilon Q_t =$$

Can we recover Q_t from $B_\varepsilon Q_t$?

Claim: the operator $B_\varepsilon : \mathbb{B}(S,\mathcal{B}) \to \mathbb{B}(S,\mathcal{B})$ is invertible $\forall$ small $\varepsilon > 0$.

To see why, we employ the geometric series: let $T_\varepsilon = I - B_\varepsilon$ and define $C_\varepsilon = \sum_{n=0}^{\infty} T_\varepsilon^n$ — provided this sum converges.

$$\therefore \; C_\varepsilon B_\varepsilon = \lim_{N\to\infty} \sum_{n=0}^{N} T_\varepsilon^n B_\varepsilon$$

$$= \lim_{N\to\infty} \left(I - T_\varepsilon^{N+1} \right) = I.$$

So: when does the geometric series defining C_ε converge? If $\|T_\varepsilon\|_{op} < 1$, then

$$\sum_{n=0}^{\infty} \|T_\varepsilon^n\|_{op}$$

By the Weierstrass M-test, B_ε is invertible with inverse C_ε provided $\|I - B_\varepsilon\|_{op} < 1$.

$$\| I - B_\varepsilon \|_{op} = \sup_{\|\hat{f}\|_\infty = 1} \| \underbrace{\hat{f} - B_\varepsilon \hat{f}}_{} \|_\infty$$

$$= \| \hat{f} - \frac{1}{\varepsilon} \int_0^\varepsilon Q_s \hat{f} \, ds \|_\infty$$

$$\therefore \quad \leq \frac{1}{\varepsilon} \int_0^\varepsilon \| I - Q_s \|_{op} \, ds$$

By the Mean Value Theorem for integrals, this equals

Thus, for all small $\varepsilon > 0$, $\| I - B_\varepsilon \|_{op} < 1$, and so B_ε is invertible with inverse $C_\varepsilon = \sum_{n=0}^{\infty} (I - B_\varepsilon)^n$.

$$\frac{Q_{t+h} - Q_t}{h} = \qquad C_\varepsilon \left(\frac{B_\varepsilon Q_{t+h} - B_\varepsilon Q_t}{h} \right)$$

In summary: if $\|Q_t - I\|_{op} \to 0$ as $t \downarrow 0$, then

$t \mapsto Q_t$ is operator norm continuous on $[0,\infty)$,

$A = \lim_{t \downarrow 0} \frac{Q_t - I}{t}$ exists and is bounded on $\mathbb{B}(S,\mathcal{B})$,

and $t \mapsto Q_t f$ is diff'ble on $(0,\infty)$, satisfying

$$\frac{d}{dt} Q_t = AQ_t = Q_t A \quad , \quad Q_0 = I .$$

This first-order ODE has a unique solution:

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

which converges (locally uniformly in t) in operator norm, because A is bounded.

It is generally too strong to expect $\|Q_t - I\|_{op} \to 0$ and for A to be bounded (or even defined everywhere). But these conditions do make sense for discrete state spaces.