Theorem: Let $(Q_t)_{t\geq 0}$ be Markov transition operators over $(S,\mathcal{B})$. Suppose $t\mapsto Q_t$ is operator norm continuous @ $t=0$:

$$\lim_{t\downarrow 0} \|Q_t - I\|_{op} = 0.$$

Then $t\mapsto Q_t$ is operator norm differentiable on $[0,\infty)$.
Let $A := \frac{d}{dt} Q_t \big|_{t=0^+}$. Then $\|A\|_{op} < \infty$, and

$$Q_t = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \quad \text{(which converges unif. in op. norm).}$$

In particular, Q_t satisfies the Kolmogorov forward and backward ODEs:

$$\frac{d}{dt} Q_t = Q_t A = A Q_t, \quad Q_0 = I.$$

Eg. For a Poisson process $S = \mathbb{N}$, $q_t(i,j) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \mathbb{1}_{j\geq i}$.

$$\therefore \|q_t - I\|_\infty = \sup_i \sum_{j=i}^{\infty} |q_t(i,j) - \delta_{ij}| = \underbrace{|e^{-\lambda t} - 1| + \sum_{j>i} e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}}$$

$$\sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t}(e^{\lambda t} - 1)$$

$$\| \quad 2(1-e^{-\lambda t}) \to 0 \text{ as } t\downarrow 0.$$

Pf. First, note that $\|Q_t\|_{op} \underset{=}{\leq} 1$ $\left(\|Q_t f\|_\infty \leq \|f\|_\infty \,;\; Q_t 1 = 1\right)$. $\forall t \geq 0$.

Let $t \geq 0$. If $h > 0$,

$$\|Q_{t+h} - Q_t\|_{op} = \|Q_h Q_t - Q_t\|_{op} = \|(Q_h - I) Q_t\|_{op} \leq \|Q_h - I\|_{op} \underbrace{\|Q_t\|_{op}}_{\leq 1} \underset{h \downarrow 0}{\to} 0.$$

Similarly, if $t > h$,

$$\|Q_{t-h} - Q_t\|_{op} = \|Q_{t-h} - Q_h Q_{t-h}\|_{op} \leq \|I - Q_h\|_{op} \overbrace{\|Q_{t-h}\|_{op}}^{\leq 1} \to 0 \text{ as } h \downarrow 0.$$

This shows $t \mapsto Q_t$ is operator norm continuous on $[0,\infty)$.

Similarly: $$\frac{Q_{t+h} - Q_t}{h} = \frac{Q_h - I}{h} Q_t = Q_t \frac{Q_h - I}{h}.$$

Thus, for any $t \geq 0$, and any bounded op. B,

$$\left\| \frac{Q_{t+h} - Q_t}{h} - Q_t B \right\|_{op} = \left\| Q_t \left(\frac{Q_h - I}{h} - B \right) \right\|_{op}$$
$$\leq \left\| \frac{Q_h - I}{h} - B \right\|_{op}.$$

This shows $\left(\frac{d}{dt}\right)_+ Q_t$ exists at any pt. t iff $A = \frac{d}{dt} Q_t \Big|_{t=0^+}$ exists, in which case $= Q_t A = A Q_t$.

Thus, to show $A = \frac{d}{dt} Q_t \big|_{t=0^+}$ exists

it suffices to show that $t \mapsto Q_t$ is (right) diff'ble at some $t \geq 0$.

To prove this, we employ a trick due to Lars Gårding.

Gårding's trick: For $\varepsilon > 0$, define an operator B_ε on $B(S, \mathcal{B})$ by

$$B_\varepsilon = \frac{1}{\varepsilon} \int_0^\varepsilon Q_s \, ds \qquad B_\varepsilon f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon Q_s f(x) ds$$

Note: $B_\varepsilon Q_t f = \frac{1}{\varepsilon} \int_0^\varepsilon Q_s (Q_t f) ds = \frac{1}{\varepsilon} \int_0^\varepsilon Q_{s+t} f \, ds$

I.e. $B_\varepsilon Q_t = \frac{1}{\varepsilon} \int_0^\varepsilon Q_{s+t} \, ds = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Q_u \, du$.

It follows by the Fundamental Theorem of Calculus that $t \mapsto B_\varepsilon Q_t$ is differentiable, and

$$\frac{d}{dt} B_\varepsilon Q_t = \frac{1}{\varepsilon} \left[Q_{t+\varepsilon} - Q_t \right].$$

Can we recover Q_t from $B_\varepsilon Q_t$?

Claim: the operator $B_\varepsilon : \mathbb{B}(S,\mathcal{B}) \to \mathbb{B}(S,\mathcal{B})$ is invertible $\forall$ small $\varepsilon > 0$.

To see why, we employ the geometric series: let $T_\varepsilon = I - B_\varepsilon$ and define $C_\varepsilon = \sum_{n=0}^{\infty} T_\varepsilon^n$ — provided this sum converges.

$$\therefore C_\varepsilon B_\varepsilon = \lim_{N\to\infty} \sum_{n=0}^{N} \underbrace{T_\varepsilon^n B_\varepsilon}_{\substack{T_\varepsilon^n (I - T_\varepsilon) \\ T_\varepsilon^n - T_\varepsilon^{n+1}}}$$

$$= \lim_{N\to\infty} (I - T_\varepsilon^{N+1}) = I. \qquad (B_\varepsilon C_\varepsilon = I \text{ also}).$$

So: when does the geometric series defining C_ε converge? If $\|T_\varepsilon\|_{op} < 1$, then

$$\sum_{n=0}^{\infty} \|T_\varepsilon^n\|_{op} \leq \sum_{n=0}^{\infty} \|T_\varepsilon\|_{op}^n = \frac{1}{1 - \|T_\varepsilon\|_{op}} < \infty.$$

By the Weierstrass M-test, B_ε is invertible with inverse C_ε provided $\|I - B_\varepsilon\|_{op} < 1$.

$$\| I - B_\varepsilon \|_{op} = \sup_{\|\hat{f}\|_\infty = 1} \underbrace{\| \hat{f} - B_\varepsilon \hat{f} \|_\infty}$$

$$= \| \hat{f} - \frac{1}{\varepsilon}\int_0^\varepsilon Q_s \hat{f}\, ds \|_\infty = \| \frac{1}{\varepsilon}\int_0^\varepsilon [\hat{f} - Q_s \hat{f}]\, ds \|_\infty \leq \frac{1}{\varepsilon}\int_0^\varepsilon \underbrace{\| \hat{f} - Q_s \hat{f} \|_\infty}_{\leq \|I - Q_s\|_{op}} ds$$

$$\therefore \leq \frac{1}{\varepsilon}\int_0^\varepsilon \| I - Q_s \|_{op}\, ds = \| I - Q_{s^*} \|_{op} \text{ for some } 0 \leq s^* \leq \varepsilon .$$

cont. fn of s

By the Mean Value Theorem for integrals, this equals

$\to 0$ as $\varepsilon \downarrow 0$.

Thus, for all small $\varepsilon > 0$, $\|I - B_\varepsilon\|_{op} < 1$, and so B_ε is invertible with inverse $C_\varepsilon = \sum_{n=0}^{\infty} (I - B_\varepsilon)^n$.

$$\therefore \lim_{h \to 0} \frac{Q_{t+h} - Q_t}{h} = \lim_{h\to 0} C_\varepsilon \left(\frac{B_\varepsilon Q_{t+h} - B_\varepsilon Q_t}{h} \right)$$

$$= \frac{d}{dt} Q_t \qquad = C_\varepsilon \frac{d}{dt} B_\varepsilon Q_t .$$

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In summary: if $\|Q_t - I\|_{op} \to 0$ as $t \downarrow 0$, then

$t \mapsto Q_t$ is operator norm continuous on $[0,\infty)$,

$A = \lim_{t \downarrow 0} \frac{Q_t - I}{t}$ exists and is bounded on $\mathbb{B}(S,\mathcal{B})$,

and $t \mapsto Q_t f$ is diff'ble on $(0,\infty)$, satisfying

$$\frac{d}{dt} Q_t = A Q_t = Q_t A \quad , \quad Q_0 = I .$$

This first-order ODE has a unique solution:

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

which converges (locally uniformly in t) in operator norm, because A is bounded.

It is generally too strong to expect $\|Q_t - I\|_{op} \to 0$ and for A to be bounded (or even defined everywhere). But these conditions do make sense for discrete state spaces.