Generators and (Uniform) Continuity

Given Markov transition operators $(Q_t)_{t\geq 0}$ on $\mathbb{B}(S,\mathcal{B})$, the generator (should it exist) is the linear operator A on $\mathbb{B}(S,\mathcal{B})$

$$Af = \frac{d}{dt} Q_t f \Big|_{t=0^+} = \lim_{t \downarrow 0} \frac{Q_t f - f}{t} \qquad Q_t = e^{tA}$$

If Q_t is differentiable @ $t=0$, it must be continuous @ 0.
There are many possible notions of continuity we could demand.
The strongest one, operator norm continuity, will lead to the nicest results.

Def: Given a normed space $(\mathbb{B}, \| \ \|) = (\mathbb{B}(S,\mathcal{B}), \|\cdot\|_\infty)$ and a linear operator $A: \mathbb{B} \to \mathbb{B}$, its operator norm $\|A\|_{op}$ is defined to be

$$\|A\|_{op} := \sup_{f \neq 0} \frac{\|Af\|}{\|f\|} = \sup_{\|\hat{f}\|=1} \|A\hat{f}\|$$

If $\|A\|_{op} < \infty$, A is bounded.

$\|Af - Ag\|$
$= \|A(f-g)\|$
$\leq \|A\|_{op} \|f-g\|$
A Lipschitz.

Lemma: If A, B are bounded linear operators. Then the composition AB is also bounded, and $\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}$.

Pf. If $f \neq 0$, $\dfrac{\|ABf\|}{\|f\|} = \begin{cases} 0 & \text{if } Bf = 0 \\ \dfrac{\|ABf\|}{\|Bf\|} \cdot \dfrac{\|Bf\|}{\|f\|} & \text{if } Bf \neq 0. \end{cases}$

$$\sup_{g \neq 0} \frac{\|Ag\|}{\|g\|} = \|A\|_{op}$$

$$\rotatebox{90}{$\leq$}$$

$$\therefore \sup_{f \neq 0} \frac{\|ABf\|}{\|f\|} = \sup_{Bf \neq 0} \left(\frac{\|ABf\|}{\|Bf\|} \cdot \frac{\|Bf\|}{\|f\|} \right) \leq \sup_{Bf \neq 0} \frac{\|A(Bf)\|}{\|Bf\|} \cdot \sup_{Bf \neq 0} \frac{\|Bf\|}{\|f\|} \leq \sup_{f \neq 0} \frac{\|Bf\|}{\|f\|} = \|B\|_{op}.$$

///

Cor: If A is bounded, so is A^n, and $\|A^n\|_{op} \leq \|A\|_{op}^n$.

Moreover, $e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ converges to a bounded operator, and $\|e^{tA}\|_{op} \leq e^{|t|\|A\|_{op}}$.

Pf. $\|A^n\|_{op} \leq \|A\|_{op}^n$ by induction on the Lemma.

$(\mathbb{B}(S, \mathcal{B}), \|\ \|_\infty)$ is a Banach space, so the second claim follows from the Weierstrass M-test:

$$\sum_{n=0}^{\infty} \left\|\frac{t^n}{n!} A^n\right\|_{op} \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} \|A\|_{op}^n = e^{|t|\|A\|_{op}} < \infty.$$

///

E.g. If S is countable (and $\mathcal{B} = 2^S$), we can produce operators on $\mathbb{B}(S)$ through matrices: $a: S \times S \to \mathbb{C}$, defining

$$Af(i) = \sum_{j \in S} a(i,j) f(j).$$

If S is infinite, we need some conditions on a so that this sum makes sense, and produces a new function $Af \in \mathbb{B}(S)$.

↳ Make sure $\sum_{j \in S} |a(i,j)|$ is finite, and uniformly bounded in i.

Define $\|a\|_\infty := \sup_{i \in S} \sum_{j \in S} |a(i,j)|$. If this is $< \infty$, then for any $f \in \mathbb{B}(S)$,

$$\|Af\|_\infty = \sup_i \Big| \sum_j a(i,j) f(j) \Big| \leq \sup_i \sum_j |a(i,j)| \underbrace{|f(j)|}_{\leq \|f\|_\infty}$$

$$\leq \|a\|_\infty \|f\|_\infty.$$

$$< \infty.$$

So $Af \in \mathbb{B}(S)$.

In fact, $\|a\|_\infty = \|A\|_{op}$.

Prop: Let S be countable, and $a : S\times S \to \mathbb{C}$. If $\|a\|_\infty < \infty$, then

$$(Af)(i) = \sum_{j\in S} a(i,j) f(j)$$

defines a bounded linear operator on $\mathbb{B}(S)$, and $\|A\|_{op} = \|a\|_\infty$.

Pf. We showed on the previous slide that $\|Af\|_\infty \leq \|a\|_\infty \|f\|_\infty$.

Conversely, for each $i\in S$, let $f_i \in \mathbb{B}(S)$ be given by

$\|f_i\|_\infty = 1$. $\to$ $f_i(j) = \mathrm{sgn}^*(a(i,j)) = \begin{cases} \overline{a(i,j)}/|a(i,j)| & \text{if } a(i,j)\neq 0 \\ 1 & \text{if } a(i,j) = 0 \end{cases}$.

Then $|(Af_i)(i)| = \sum_{j\in S} a(i,j) f_i(j)$

$\|Af\|_\infty / \|f\|_\infty$

$= \sum_j |a(i,j)|$

$\sup_j |Af_i(j)| \geq$ $\forall i$

$\therefore$ $\|A\|_{op} \geq \sup_i \sum_j |a(i,j)| = \|a\|_\infty$. ///

Caution: Not every bounded linear operator on $\mathbb{B}(S)$ has a matrix!

Theorem: Let $(Q_t)_{t\geq 0}$ be Markov transition operators over $(S, \mathcal{B})$.
Suppose $t \mapsto Q_t$ is operator norm continuous @ $t=0$:

$$\lim_{t \downarrow 0} \| Q_t - I \|_{op} = 0.$$

Then $t \mapsto Q_t$ is operator norm differentiable on $[0, \infty)$.
Let $A := \frac{d}{dt} Q_t \big|_{t=0^+} = \lim_{t \downarrow 0} \frac{1}{t} [Q_t - I]$. Then $\|A\|_{op} < \infty$, and

$$Q_t = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

In particular, Q_t satisfies the Kolmogorov forward and backward ODEs:

$$\frac{d}{dt} Q_t = Q_t A = A Q_t, \quad Q_0 = I.$$

Remarks: 1. Using power-series methods, it's standard to check e^{tA} is the unique sol'n

2. Without op. norm continuity, A might still exist, but may be unbounded / map into unbounded functions.