In the continuous (homogeneous) time case, the Chapman-Kolmogorov equations:

$$Q_{t+s} = Q_t Q_s \ , \quad Q_0 = I$$

The semigroup property is very suggestive of the exponential.

E.g. If the state space is finite $S = \{1, 2, \ldots, n\}$, if there is some matrix $A \in \mathbb{M}_{n\times n}$ s.t. $Q_t = e^{tA}$ then $Q_0 = I$ and

$$Q_{t+s} = e^{(t+s)A}$$

How can we find A if it exists?

Note: $Q_t(i,j) = \mathbb{P}(X_t = j \mid X_0 = i)$ and so we have

$$\geq 0 \ \forall i,j, \quad \sum_j Q_t(i,j) = 1 \ \forall i.$$

What does this say about A?

$$A_{ij} = \frac{d}{dt} Q_t \Big|_{t=0} = \lim_{t \downarrow 0} \frac{Q_t(i,j) - \delta_{ij}}{t}$$

Prop. Let $A \in M_{n\times n}$. Then $Q_t = e^{tA}$ is a time-homogeneous Markov transition operator iff $A_{ij} \geq 0$ $1 \leq i \neq j \leq n$ and $\forall i$ $\sum_{j=1}^{n} A_{ij} = 0$.

Pf. ($\Longrightarrow$) Previous slide.

($\Longleftarrow$) HW.

Now, is it always true that $Q_t = e^{tA}$ for some operator A?

E.g. Fix a probability mass vector $\vec{v}$, and take $P = \begin{bmatrix} \vec{v} \\ \vdots \\ \vec{v} \end{bmatrix}$

Then $q_t(i,j) = P_{ij}$ $\forall t > 0$ is a semigroup!

But this is not continuous!

$\lim_{t \to 0} Q_t = P \neq I$.

What does this mean about the process?

$P(X_t = j \mid X_s = i) = q_{t-s}(i,j) = v(j)$

Eg. Poisson process revisited.

$$q_{s,t}(m,n) = \mathbb{P}(N_t = n \mid N_s = m)$$
$$= \mathbb{P}(N_t - N_s + N_s = n \mid N_s = m)$$

$$= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-m}}{(n-m)!},$$

$$\therefore (Q_t f)(k) = \sum_{n \in \mathbb{N}} q_t(k,n) f(n)$$

Is there a generator A? If so

$$(Af)(k) = \frac{d}{dt} Q_t f(k) \Big|_{t=0}$$

$$Q_t f(k) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n)$$

$$\therefore \frac{d}{dt} Q_t f(k) = -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n) + e^{-\lambda t} \sum_{n=0}^{\infty} \lambda \cdot \frac{n(\lambda t)^{n-1}}{n!} f(k+n)$$

$$= -\lambda Q_t f(k) + \lambda Q_t f(\cdot + 1)$$

I.e. set $(Af)(k) = \lambda[f(k+1) - f(k)]$. Thus $\frac{d}{dt} Q_t f = Q_t A f = A Q_t f$.

$\frac{d}{dt} Q_t = A Q_t$ should mean $Q_t = e^{tA}$

In this case, we can make literal sense of this by power series: $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$

$$(Af)(k) = \lambda[f(k+1) - f(k)]$$
$$= \lambda[(Tf)(k) - f(k)] = \lambda(T-I)f(k).$$

T commutes with $-I$

$$\therefore e^{tA} = e^{\lambda t(T-I)} = e^{-\lambda t I} e^{\lambda t T}$$

$$(e^{\lambda t T} f)(k) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} T^n f(k)$$

$$\therefore (e^{tA} f)(k) = e^{-\lambda t} e^{\lambda t T} f(k) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f(k+n)$$

Thus, A <u>is</u> the generator of the Markov process $(N_t)_{t \geq 0}$.

$$(Af)(k) = \lambda[f(k+1) - f(k)]$$

$\mathbb{B}(\mathbb{N})$ has "basis" $\{\mathbb{1}_k : k \in \mathbb{N}\}$.

In terms of this basis, the matrix of A is

E.g. Brownian motion B_t.

$$q_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t} \qquad Q_t f(x) = \int_{\mathbb{R}} q_t(x,y) f(y)\, dy$$

Is there a generator A? I.e. $Q_t f = A Q_t f = Q_t A f \quad \forall f \in B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

If so, $A = \frac{d}{dt} Q_t f \big|_{t=0}$

Prop: If $f \in C_c^2(\mathbb{R})$ then

$$\frac{d}{dt} Q_t f(x) = A Q_t f(x) = Q_t A f(x)$$

where $Af(x) = \frac{1}{2} f''(x)$.

"The generator of Brownian motion is $\frac{1}{2}\frac{d^2}{dx^2}$."

Problem: $\nexists\, A : B(\mathbb{R}, \mathcal{B}(\mathbb{R})) \to B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $A\big|_{C_c^2(\mathbb{R})} = \frac{1}{2}\frac{d^2}{dx^2}$.