In the continuous (homogeneous) time case, the Chapman-Kolmogorov equations:

$$Q_{t+s} = Q_t Q_s \quad , \quad Q_0 = I$$

The semigroup property is very suggestive of the exponential.

E.g. If the state space is finite $S = \{1, 2, \ldots, n\}$, if there is some matrix $A \in M_{n \times n}$ s.t. $Q_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ then $Q_0 = I$ and

Markov generator

$$Q_{t+s} = e^{(t+s)A} = e^{tA} e^{sA} = Q_t Q_s .$$

$e^{A+B} \neq e^A e^B$ unless $AB = BA$

How can we find A if it exists? $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A \quad \therefore A = \frac{d}{dt} Q_t \Big|_{t=0}$.

Note: $Q_t(i,j) = \mathbb{P}(X_t = j \mid X_0 = i)$ and so we have

$$\geq 0 \;\; \forall i, j, \qquad \sum_j Q_t(i,j) = 1 \;\; \forall i .$$

What does this say about A?

$$\underline{A_{ij}} = \frac{d}{dt}[Q_t]_{ij}\Big|_{t=0} = \lim_{t \downarrow 0} \frac{Q_t(i,j) - \delta_{ij}}{t}$$

If $i \neq j$, $\frac{Q_t(i,j)}{t} \geq 0$

$$\sum_j A_{ij} = \lim_{t \downarrow 0} \frac{1}{t} \sum_j [Q_t(i,j) - \delta_{ij}] = \lim_{t \downarrow 0} \frac{1}{t}\Big[\big(\sum_j Q_t(i,j)\big) - 1\Big] = 0$$

Prop. Let $A \in M_{n\times n}$. Then $Q_t = e^{tA}$ is a time-homogeneous Markov transition operator iff $A_{ij} \geq 0$ $1 \leq i \neq j \leq n$ and $\forall i$ $\sum_{j=1}^{n} A_{ij} = 0$.

Pf. ($\Longrightarrow$) Previous slide.

($\Longleftarrow$) HW.

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Now, is it always true that $Q_t = e^{tA}$ for some operator A?

E.g. Fix a probability mass vector $\vec{v}$, and take $P = \begin{bmatrix} \vec{v} \\ \vdots \\ \vec{v} \end{bmatrix}$ $P_{ij} = v(j)$ $\forall i$.

Then $q_t(i,j) = P_{ij}$ $\forall t > 0$ is a semigroup!

$$\sum_k q_s(i,k)\, q_t(k,j) = \sum_k P_{ik} \underbrace{P_{kj}}_{v(j)} = v(j) \sum_{k} \cancel{P_{ik}} = P_{ij} = q_{s+t}(i,j).$$

But this is not continuous!

$\lim_{t \downarrow 0} Q_t = P \neq I$.

What does this mean about the process?

$\mathbb{P}(X_t = j \mid X_s = i) = q_{t-s}(i,j) = v(j) = \mathbb{P}(X_t = j)$.

$\therefore$ $\{X_t\}_{t \geq 0}$ are i.i.d. with law v.

Eg. Poisson process revisited.

$$
\begin{aligned}
q_{s,t}(m,n) &= \mathbb{P}(N_t = n \mid N_s = m) \\
&= \mathbb{P}(N_t - N_s + N_s = n \mid N_s = m) \\
&= \mathbb{P}(N_t - N_s + m = n \mid N_s = m) \\
&= \mathbb{P}(N_t - N_s = n-m) \\
&= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-m}}{(n-m)!},
\end{aligned}
$$

$N_t - N_s$, $N_s - \underset{\overset{\parallel}{0}}{N_0}$ indep.

$$
\begin{aligned}
\therefore (Q_t f)(k) &= \sum_{n \in \mathbb{N}} q_t(k,n) f(n) \\
&= \sum_{n \geq k} e^{-\lambda t} \frac{(\lambda t)^{n-k}}{(n-k)!} f(n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n}}{n!} f(k+n).
\end{aligned}
$$

Is there a generator A? If so

$$
(Af)(k) = \frac{d}{dt} Q_t f(k) \Big|_{t=0}
$$

$$Q_t f(k) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n)$$ $\leftarrow$ differentiable power series; converges for $t \in \mathbb{R}$ b/c $f \in \mathcal{B}(\mathbb{N})$.

$$\therefore \frac{d}{dt} Q_t f(k) = -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n) + e^{-\lambda t} \sum_{n=1}^{\infty} \lambda \cdot \frac{n(\lambda t)^{n-1}}{n!} f(k+n)$$

$$\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n+1)$$

$$= -\lambda Q_t f(k) + \lambda Q_t f(\cdot + 1)$$

I.e. set $(Af)(k) = \lambda[f(k+1) - f(k)]$. Thus $\frac{d}{dt} Q_t f = Q_t A f = A Q_t f$.

$\frac{d}{dt} Q_t = A Q_t$ should mean $Q_t = e^{tA}$

and $Q_0 = I$

In this case, we can make literal sense of this by power series: $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \overset{?}{=} Q_t$

If $AB = BA$ $\quad e^{t(A+B)} = e^{tA} e^{tB}$.

$$(Af)(k) = \lambda[f(k+1) - f(k)]$$
$$= \lambda[(Tf)(k) - f(k)] = \lambda(T-I)f(k).$$

T commutes with $-I$

$$\therefore e^{tA} = e^{\lambda t(T-I)} = \underbrace{e^{-\lambda t I}}_{e^{-\lambda t},} e^{\lambda t T}$$

$$(e^{\lambda t T} f)(k) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} T^n f(k)$$

$Tf(k) = f(k+1)$

$(T^2 f)(k) = Tf(k+1)$
$= f(k+2)$
$\vdots$
$T^n f(k) = f(k+n)$

$$= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f(k+n)$$

$$\therefore (e^{tA} f)(k) = e^{-\lambda t} e^{\lambda t T} f(k) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f(k+n) = Q_t f(k).$$

Thus, A is the generator of the Markov process $(N_t)_{t \geq 0}$.

$$(Af)(k) = \underset{\text{rate } \lambda.}{\lambda}[\underbrace{f(k+1) - f(k)}_{\text{jump of size } 1}]$$

$$[A] = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 \\ 0 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

$\mathbb{B}(\mathbb{N})$ has "basis" $\{\mathbb{1}_k : k \in \mathbb{N}\}$.

In terms of this basis, the matrix of A is

$$-\lambda I + \lambda \underset{\text{shift}}{[T]}.$$

E.g. Brownian motion B_t.

"heat kernel" $\nearrow$ $q_t(x,y) = \underbrace{\frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t}}_{\gamma_t(x-y).}$

$$Q_t f(x) = \int_{\mathbb{R}} q_t(x,y) f(y)\,dy = (\gamma_t * f)(x).$$

Is there a generator A? I.e. $\frac{d}{dt} Q_t f = A Q_t f = Q_t A f \quad \forall f \in B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

If so, $A = \frac{d}{dt} Q_t f \big|_{t=0}$ $\therefore$ it's unique.

"Feller Processes"

Prop: If $f \in C_c^2(\mathbb{R}) \subsetneq B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then

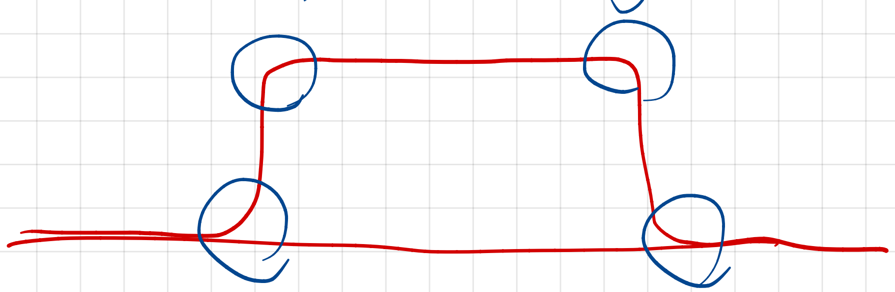
$$\frac{d}{dt} Q_t f(x) = A Q_t f(x) = Q_t A f(x)$$

where $Af(x) = \frac{1}{2} f''(x)$.

"The generator of Brownian motion is $\frac{1}{2}\frac{d^2}{dx^2}$."

Problem: $\nexists\, A : B(\mathbb{R}, \mathcal{B}(\mathbb{R})) \to B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $A|_{C_c^2(\mathbb{R})} = \frac{1}{2}\frac{d^2}{dx^2}$.

$\mathbb{1}_{[0,1]} = \lim_{n\to\infty} \varphi_n$



$\|\varphi_n''\|_\infty \to \infty$.