Discrete Time Homogeneous Processes

$T = \mathbb{N} = \{0, 1, 2, \dots\}$

Chapman-Kolmogorov equations: $Q_{n+m} = Q_n Q_m$.

$\therefore \quad Q_n = Q_1 Q_{n-1}$

The dynamics is described by iterating a single transition operator

$$Q_1 : \mathbb{B}(S, \mathcal{B}) \to \mathbb{B}(S, \mathcal{B})$$

Eg. Random walk $X_n = \sum_{k=1}^{n} \xi_k$, $\leftarrow$ $\{\xi_k\}_{k=1}^{\infty}$ iid random vectors.

$$(Q_1 f)(x) = \mathbb{E}[f(x + X_1 - 0)]$$

$\hookrightarrow$ If $S = \mathbb{Z}$, $\xi_k \overset{d}{=} p\delta_1 + (1-p)\delta_{-1}$

In discrete time and discrete (state) space:

$$(X_n)_{n\in\mathbb{N}} \qquad X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, 2^S)$$

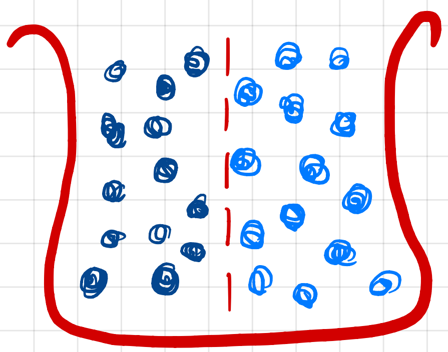
$$q_{n,m}(x, B) = \sum_{y\in B} q_{n,m}(x,y)$$

$$q_{n,m}(x,y) = \mathbb{P}(X_m = y \mid X_n = x)$$

Eg. (Ehrenfest Urn)

Model as a Markov process:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} & |i-j|>1 \\ & j = i-1 \\ & j = i+1 \end{cases}$$



Semipermeable membrane, N particles total.

X_n = # particles on left.

At each time, choose a particle uniformly at random from the whole urn, and move it to the other side of the membrane.

If $(X_t)_{t\in T}$ is a Markov process taking values in a discrete state space, it is typically called a **Markov Chain**.

(Some authors also call a discrete time process $(X_n)_{n\in\mathbb{N}}$ a Markov Chain for any state space - discrete or not. Everyone agrees that $(X_t)_{t\geq 0}$ with continuous time and state space is a Markov process.)

Let's focus on discrete (homogeneous) time and space.

ν_k = probability mass function of X_k

$$q_1(i,j) = \mathbb{P}(X_1 = j \mid X_0 = i)$$

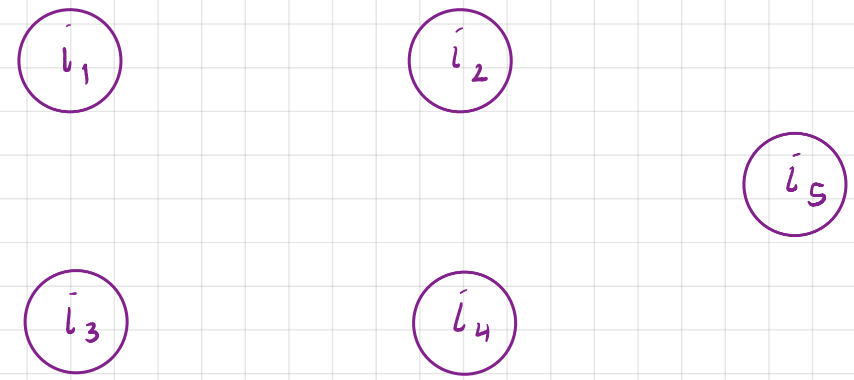
$$q_n(i,j) =$$

$$\therefore \quad \nu_n(j) = \sum_i \nu_0(i)\,\mathbb{P}(X_n = j \mid X_0 = i)$$

Note: this is only a small part of what "Markov Chain" means.

Eg. $\mathbb{P}(X_n = j, X_m = k) = \mathbb{P}(X_m = k \mid X_n = j)\,\nu_n(j)$

This is not a course on Markov chains - a rich and important field.
In the finite state space case, we often represent the data of the process in a (looped) graph:



An arrow $i \xrightarrow{P} j$ means

Note: for each $i \in S$, $\sum_{j \in S} \mathbb{P}(X_1 = j \mid X_0 = i)$

I.e. the Markov matrix P is a stochastic matrix

$$\forall i \qquad \sum_j P_{ij} = 1$$