Discrete Time Homogeneous Processes

$T = \mathbb{N} = \{0, 1, 2, \dots\}$

Chapman-Kolmogorov equations: $Q_{n+m} = Q_n Q_m$.

$\therefore \; Q_n = Q_1 Q_{n-1} = Q_1 Q_1 Q_{n-2} \cdots Q_1^n$

The dynamics is described by iterating a single transition operator

$$Q_1 : \mathbb{B}(S, \mathcal{B}) \to \mathbb{B}(S, \mathcal{B})$$

E.g. Random walk $X_n = \sum_{k=1}^{n} \xi_k$, $\leftarrow \{\xi_k\}_{k=1}^{\infty}$ iid random vectors.

$$(Q_1 f)(x) = \mathbb{E}[f(x + X_1 - 0)]$$

$$= \int f(x+y)\, \mu_{\xi_1}(dy)$$

If $S = \mathbb{Z}$, $\xi_k \overset{d}{=} p\delta_1 + (1-p)\delta_{-1}$

$$(Q_1 f)(x) = p\, f(x+1) + (1-p) f(x-1)$$

In discrete time and discrete (state) space: countable

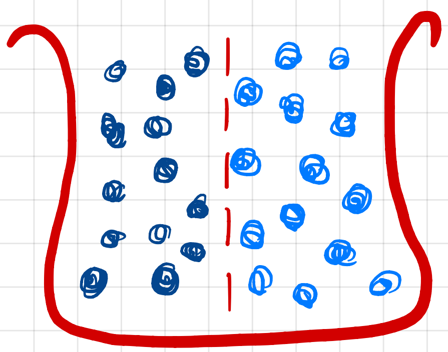
$$(X_n)_{n\in\mathbb{N}} \qquad X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, 2^S)$$

$$q_{n,m}(x,B) = \sum_{y\in B} q_{n,m}(x,y) = \sum_{\substack{y_{n+1},\dots,y_{m-1}\in S\\ y\in B}} q_{n,n+1}(x,y_{n+1})\, q_{n+1,n+2}(y_{n+1},y_{n+2}) \cdots q_{m-1,m}(y_{m-1},y)$$

$$q_{n,m}(x,y) = \mathbb{P}(X_m = y \mid X_n = x)$$

$$q_1(x,y) = \mathbb{P}(X_1 = y \mid X_0 = x)$$

Eg. (Ehrenfest Urn)



Model as a Markov process:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i)$$

$$q_{n,n+1}(i,j) = q_{0,1}(i,j) = q_1(i,j) = \begin{cases} 0 & |i-j|>1,\ i=j \\ i/N & j = i-1 \\ 1 - i/N & j = i+1 \end{cases}$$

Time Homogeneous Markov Process.

Semipermeable membrane, N particles total.

X_n = # particles on left.

At each time, choose a particle uniformly at random from the whole urn, and move it to the other side of the membrane.

If $(X_t)_{t\in T}$ is a Markov process taking values in a discrete state space, it is typically called a Markov Chain.

(Some authors also call a discrete time process $(X_n)_{n\in\mathbb{N}}$ a Markov Chain for any state space - discrete or not. Everyone agrees that $(X_t)_{t\geq 0}$ with continuous time and state space is a Markov process.)

Let's focus on discrete (homogeneous) time and space.

ν_k = probability mass function of X_k $\quad \nu_k(i) = \mathbb{P}(X_k = i)$, $i\in S$.

$$q_1(i,j) = \mathbb{P}(X_1 = j \mid X_0 = i) \quad S\times S \text{ matrix } P.$$

$$q_n(i,j) = \sum_{k_1,\dots,k_{n-1}} q_1(i,k_1)\, q_1(k_1,k_2) \cdots q_1(k_{n-1},j)$$

$$= [P^n]_{ij}$$

row vector.

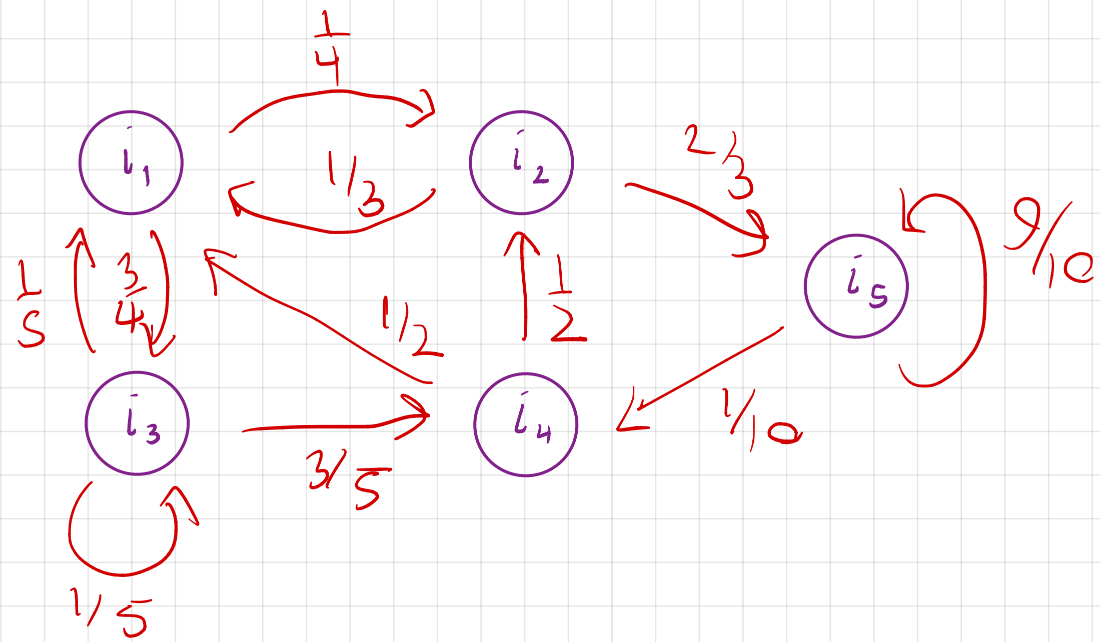
$$\therefore \; \nu_n(j) = \sum_i \nu_0(i)\, \mathbb{P}(X_n = j \mid X_0 = i) = [\vec{\nu}_0 P^n]_j$$

Note: this is only a small part of what "Markov Chain" means.

Eg. $\mathbb{P}(X_n = j, X_m = k) = \mathbb{P}(X_m = k \mid X_n = j)\, \nu_n(j) = [\vec{\nu}_0 P^n]_j \cdot [P^{m-n}]_{jk}$.

This is not a course on Markov chains - a rich and important field.
In the finite state space case, we often represent the data of the process in a (looped) graph:

$q_1(i_3, i_1) = \frac{1}{5}$ $\longrightarrow$



$\sum_{i \in S} q_1(i_3, i) = 1.$

In the time inhomogeneous setting, the arrow labels would change w/ n.

An arrow $i \xrightarrow{p} j$ means $\mathbb{P}(X_{n+1} = j \mid X_n = i) = p$

Note: for each $i \in S$, $\sum_{j \in S} \mathbb{P}(X_1 = j \mid X_0 = i) = 1$.

$$\overset{\parallel}{\sum_j q_1(i,j)} = \sum_j P_{ij}$$

I.e. the Markov matrix P is a stochastic matrix

$$\forall i \qquad \sum_j P_{ij} = 1$$

Any stochastic matrix is the Markov matrix of a chain.