Time Homogeneity

Eg. Poisson process N_t $\quad q_{s,t}(n,B) = \mathbb{E}[\mathbb{1}_B(n+N_t-N_s)]$

Eg. (pre-) Brownian motion B_t

$$q_{s,t}(x,B) = \mathbb{E}[\mathbb{1}_B(x+B_t-B_s)]$$

These transition kernels $q_{s,t}$ depend on s,t only through $t-s$.

Def. A collection $\{Q_{s,t}\}_{s\le t\in T}$ of Markov transition operators is called **time homogeneous** if

$$Q_{s,t} = Q_{0,t-s} \qquad \forall s\le t\in T.$$

In this case, the Chapman-Kolmogorov equations become $\quad Q_sQ_t = Q_{s+t}, \quad Q_0 = \mathrm{Id}.$

A collection of operators $(Q_t)_{t\in T}$ for which $Q_s Q_t = Q_{s+t}$, $Q_0 = Id$.
is called a **1-parameter semigroup**.

If $(X_t)_{t\in T}$ is a Markov process with time homogeneous transition operators, it is called a time **homogeneous Markov process**.

$$\mathbb{E}[f(X_t)\,|\,\mathcal{F}_s] = (Q_{t-s}f)(X_s)$$

Heuristic:
Q_t determines the process at time t (given X_0).

Q_{t+s} $\qquad\qquad\qquad\qquad$ $Q_t Q_s$

X_{t+s} given X_0 $\qquad\qquad\qquad\qquad$ X_s given X_0

For any time homogeneous Markov process, all f.d. distributions are determined by ν_0 and the transition semigroup $(Q_t)_{t\in T}$.

Thinking of the process as a measure on path space, we fix the transition semigroup and consider the family $\mathbb{P}^\nu$ of processes with different starting distributions ν.

Notation: We let $X_\cdot = (X_t)_{t\in T}$ denote a whole family of Markov processes with given transition semigroup $(Q_t)_{t\in T}$. For $F \in \mathbb{B}(S^T, \mathcal{B}^{\otimes T})$,

$$\mathbb{E}^\nu[F(X_\cdot)] = \text{expected value of } F(\text{the process with } X_0 \overset{d}{=} \nu).$$

In the case $\nu = \delta_x$, we write

Theorem: If $X_\cdot$ is a time homogeneous Markov process for $F \in \mathbb{B}(S^T, \mathcal{B}^{\otimes T})$, $x \mapsto \mathbb{E}^x[F(X)]$ is measurable.

Moreover, for $t \geq 0$, $\nu_0 \in \mathrm{Prob}(S, \mathcal{B})$,

$$\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid \mathcal{F}_t] = \mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid X_t] = \mathbb{E}^{X_t}[F(X)]$$

Theorem: If $X_\cdot$ is a time homogeneous Markov process, $F \in \mathcal{B}(S^{\otimes T})$

1. $S \ni x \longmapsto \mathbb{E}^x[F(X_\cdot)]$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ - measurable
2. $\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid \mathcal{F}_t] = \mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid X_t] = \mathbb{E}^{X_t}[F(X)]$

Pf. $F(X_{t+\cdot}) \in \mathcal{B}(\Omega, \mathcal{F}^X_{\geq t})$. We showed in [Lecture 36.1] that if $Y \in$ then

For the remainder of the proof, we take F of the form

$F(\omega) = f_0(\omega(t_0)) f_1(\omega(t_1)) \cdots f_n(\omega(t_n))$

Prove 1, 2 for such F; then extend by Dynkin.

1. $\mathbb{E}^x[F(X_\cdot)] = \mathbb{E}^x[f_0(X_{t_0}) \cdots f_n(X_{t_n})]$

2. Want to show $\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) | X_t] = \mathbb{E}^{X_t}[F(X)]$

$$F(X_{t+\cdot}) = f_0(X_t) f_1(X_{t+t_1}) \cdots f_n(X_{t+t_n}).$$

$\therefore$ $\forall h \in B(S, \mathcal{B})$,

$$\mathbb{E}^{\nu_0}[h(X_t) F(X_{t+\cdot})] = \mathbb{E}^{\nu_0}[h(X_t) f_0(X_t) f_1(X_{t+t_1}) \cdots f_n(X_{t+t_n})]$$

Now $\nu_0(dx) \int q_t(x, dy) = \nu_t(dx)$