Time Homogeneity

Eg. Poisson process N_t $\quad q_{s,t}(n,B) = \mathbb{E}[\mathbb{1}_B(n + \underbrace{N_t - N_s}_{\text{Poisson}(\lambda(t-s))})]$

$$= \sum_{m \in B} \text{Poisson}(\lambda(t-s))\{m-n\}$$

Eg. (pre-) Brownian motion B_t

$$q_{s,t}(x,B) = \mathbb{E}[\mathbb{1}_B(x + \underbrace{B_t - B_s}_{\mathcal{N}(0,\, t-s)})]$$

$\Big\}\ q_{s,t} = q_{0,t-s}$

These transition kernels $q_{s,t}$ depend on s,t only through $t-s$.

Def. A collection $\{Q_{s,t}\}_{s \le t \in \mathbb{T}}$ of Markov transition operators is called *time homogeneous* if

$$Q_{s,t} = Q_{0,t-s} =: Q_{t-s}. \quad \forall s \le t \in \mathbb{T}.$$

In this case, the Chapman-Kolmogorov equations become $\quad Q_s Q_t = Q_{s+t}, \quad Q_0 = \mathrm{Id}.$

$$Q_{r,s} Q_{s,t} = Q_{s-r} Q_{t-s} = Q_{\cancel{s}-r+t-\cancel{s}} = Q_{t-r} = Q_{r,t}.$$

A collection of operators $(Q_t)_{t\in T}$ for which $Q_s Q_t = Q_{s+t}$, $Q_0 = Id$.
is called a 1-parameter semigroup.

If $(X_t)_{t\in T}$ is a Markov process with time homogeneous transition operators,
it is called a time homogeneous Markov process.

$$\mathbb{E}[f(X_t)\,|\,\mathcal{F}_s] = (Q_{t-s}f)(X_s)$$

$$\mathbb{E}[f(X_{s+t})\,|\,\mathcal{F}_s] = (Q_t f)(X_s)$$

Heuristic:

Q_t determines the process at time t (given X_0).

$$Q_{t+s} = Q_t Q_s$$

X_{t+s} given X_0 $\qquad\qquad$ $\tilde{X}_t$ given X_s given X_0

new $\tilde{X}_0$

For any time homogeneous Markov process, all f.d. distributions are determined by $\nu_0 = \mathrm{Law}(X_0)$ and the transition semigroup $(Q_t)_{t\in T}$.

Thinking of the process as a measure on path space, we fix the transition semigroup and consider the family $\mathbb{P}^\nu$ of processes with different starting distributions ν.

Notation: We let $X_\cdot = (X_t)_{t\in T}$ denote a whole family of Markov processes with given transition semigroup $(Q_t)_{t\in T}$. For $F \in \mathbb{B}(S^T, \mathcal{B}^{\otimes T})$

$$\mathbb{E}^\nu[F(X_\cdot)] = \text{expected value of } F(\text{the process with } X_0 \overset{d}{=} \nu).$$

In the case $\nu = \delta_x$, we write $\mathbb{E}^x$, $\mathbb{P}^x$.

Theorem: If $X_\cdot$ is a time homogeneous Markov process for $F \in \mathbb{B}(S^T, \mathcal{B}^{\otimes T})$, $x \mapsto \mathbb{E}^x[F(X)]$ is measurable.

Moreover, for $t \geq 0$, $\nu_0 \in \mathrm{Prob}(S, \mathcal{B})$, $\mathbb{E}^x[F(X_\cdot)] = g(x)$

$$\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid \mathcal{F}_t] = \mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid X_t] = \underbrace{\mathbb{E}^{X_t}[F(X)]}_{g(X_t)}$$

Theorem: If $X_\cdot$ is a time homogeneous Markov process, $F \in \mathbb{B}(S^{\otimes T})$

1. $S \ni x \longmapsto \mathbb{E}^x[F(X_\cdot)]$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ - measurable $\leftarrow$ Does not require time homog.
2. $\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid \mathcal{F}_t] \overset{\checkmark}{=} \mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid X_t] = \mathbb{E}^{X_t}[F(X)]$

Pf. $F(X_{t+\cdot}) \in \mathbb{B}(\Omega, \mathcal{F}^X_{\geq t})$. We showed in [Lecture 36.1] that if $Y \in$ then $\mathbb{E}[Y \mid \mathcal{F}_t] = \mathbb{E}[Y \mid X_t]$. $Y = F(X_{t+\cdot})$

For the remainder of the proof, we take F of the form

$F(\omega) = f_0(\omega(t_0)) f_1(\omega(t_1)) \cdots f_n(\omega(t_n))$ $\quad 0 = t_0 < t_1 < \cdots < t_n$, $f_j \in \mathbb{B}(S, \mathcal{B})$.

Prove 1, 2 for such F; then extend by Dynkin.

1. $\mathbb{E}^x[F(X_\cdot)] = \mathbb{E}^x[f_0(X_{t_0}) \cdots f_n(X_{t_n})]$

$= \int \delta_x(dx_0) \int q_{t_0,t_1}(x_0, dx_1) \cdots \int q_{t_{n-1},t_n}(x_{n-1}, dx_n) f(x_0) \cdots f(x_n)$

$= \mathbb{E}_{\delta_x}[f_0 Q_{t_0,t_1}(\cdots f_{n-1}(Q_{t_{n-1},t_n} f_n) \cdots)]$.

$= f_0(x) Q_{t_0,t_1}(\cdots f_{n-1}(Q_{t_{n-1},t_n} f_n) \cdots)(x)$ meas. $\checkmark$

2. Want to show $\mathbb{E}^{\nu_0}[F(X_{t+\cdot}) \mid X_t] = \mathbb{E}^{X_t}[F(X)] = g(X_t)$

if $g(x) = \mathbb{E}^x[F(X_\cdot)]$

$$F(X_{t+\cdot}) = f_0(X_t) f_1(X_{t+t_1}) \cdots f_n(X_{t+t_n}).$$

$\therefore\ \forall h \in \mathbb{B}(S, \mathcal{B})$,

$$\mathbb{E}^{\nu_0}[h(X_t) F(X_{t+\cdot})] = \mathbb{E}^{\nu_0}[h(X_t) \overset{1(X_0)}{\overset{\downarrow}{f_0}}(X_t) f_1(X_{t+t_1}) \cdots f_n(X_{t+t_n})]$$

$$= \int \nu_0(dx_{-1}) \int \underset{q_t}{q_{0,t}}(x_{-1}, dx_0)\, \underset{q_{t_1}}{q_{t,t+t_1}}(x_0, dx_1) \cdots \underset{q_{t_n - t_{n-1}}}{q_{t+t_{n-1}, t+t_n}}(x_{n-1}, dx_n)\, h(x_0) f_0(x_0) \cdots f_n(x_n)$$

Now $\nu_0(dx) \int q_t(x, dy) = \nu_t(dx)$ $\quad \mathrm{Law}(X_t)$

$$= \int \nu_t(dx_0) \int q_{t_1}(x_0, dx_1) \cdots q_{t_n - t_{n-1}}(x_{n-1}, dx_n)\, h(x_0) f_0(x_0) \cdots f_n(x_n)$$

$$= \int h(x_0) \nu_t(dx_0) \underbrace{\int q_{t_1}(x_0, dx_1) \cdots q_{t_n - t_{n-1}}(x_{n-1}, dx_n)\, f_0(x_0) \cdots f_n(x_n)}_{\underbrace{\mathbb{E}^{x_0}[f_0(X_0) f_1(X_{t_1}) \cdots f_n(X_{t_n})]}_{\mathbb{E}^{x_0}[F(X_\cdot)]}}$$

$$= \mathbb{E}^{\nu_0}[h(X_t)\, \mathbb{E}^{X_t}[F(X_\cdot)]]. \leftarrow \mathbb{E}^{x_0}[F(X_\cdot)]$$

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