Given $\nu \in \mathrm{Prob}(S, \mathcal{B})$ and Markov transition operators $\{Q_{s,t}\}_{s \le t \in T}$, we constructed a Markov process as a measure on S^T:

$$X_t(\omega) = \omega(t)$$

Not every Markov process with this data $(\nu, Q_{s,t})$ "is" this one.

But every stochastic process can be identified with a measure on path space:

$$(X_t)_{t \in T} \rightsquigarrow \mathbb{P}_X \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$$

$$\mathbb{P}_X(E) =$$

$\mathbb{P}_X$ is the law of the process $t \mapsto X_t$.

Problem: $\mathcal{B}^{\otimes T} = \sigma\{\pi_t : S^T \to S\}_{t \in T}$ is small.

Eg. $C([0,\infty), \mathbb{R})$ is not in $\mathcal{B}(\mathbb{R})^{[0,\infty)}$

Brownian Motion

An adapted stochastic process $(B_t)_{t\geq 0}$ is called a (standard) Brownian motion if

1. It has independent increments
2. $B_t - B_s \overset{d}{=} \mathcal{N}(0, t-s)$ for $0 \leq s < t < \infty$
3. $t \mapsto B_t$ is continuous.

We can construct a Markov process $(X_t)_{t\geq 0}$ satisfying (1-2) by Kolmogorov:

$$q_{s,t}(x, B) =$$

provided these satisfy Chapman-Kolmogorov.

$$(Q_{s,t}f)(x) = \int_{\mathbb{R}} f(y)\, q_{s,t}(x,dy) = \int_{\mathbb{R}} f(y)\, (2\pi(t-s))^{-1/2} e^{-(x-y)^2/2(t-s)}\, dy$$

$$\therefore\ Q_{r,s} Q_{s,t} f$$

Incorporating Continuity will take some major work.
One possibility: explicit construction of B_t, like we did with the Poisson process.

$\{\xi_n\}_{n=1}^{\infty}$ iid $\mathcal{N}(0,1)$ random variables

$$B_t = \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\xi_n}{n} \sin\left(n \frac{\pi}{2} t\right)$$

Compare to Poisson process $(N_t)_{t \geq 0}$

We explicitly constructed it as

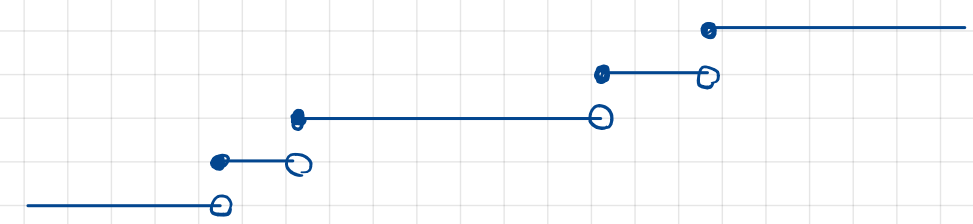
$$N_t(\omega) = \sup\{n \in \mathbb{N} : X_n \leq t\}, \text{ with } X_n = \xi_1 + \cdots + \xi_n$$

$\{\xi_k\}_{k=1}^{\infty}$ iid $\mathrm{Exp}(\lambda)$ rv's.

Properties: 1. Independent Increments

2. $N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s))$

3. $t \mapsto N_t$ is right-continuous, $\uparrow$



From 1-2:

$$q_{s,t}(n,B) = \mathbb{E}[\mathbb{1}_B(n + N_t - N_s)]$$

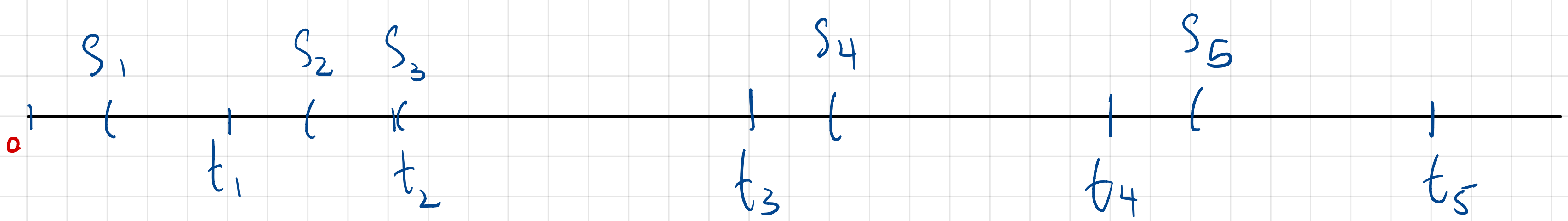
$$\begin{aligned} q_{s,t}(n,m) &= \mathbb{E}[\mathbb{1}_m(n + N_t - N_s)] \\ &= \mathbb{P}(n + N_t - N_s = m) \\ &= \mathrm{Poisson}(\lambda(t-s))\{m-n\} \end{aligned}$$

If $M_j \stackrel{d}{=} \mathrm{Poisson}(a_j)$ are independent, $M_1 + M_2 \stackrel{d}{=} \mathrm{Poisson}(a_1 + a_2)$

$\therefore$ Can verify C-K.

Prop: Let $(N_t)_{t\geq 0}$ be a process satisfying (1-2).
Set $X_n = \inf\{t \geq 0 : N_t = n\}$, and $\xi_n = X_n - X_{n-1}$, $n \geq 0$.
Then $\{\xi_n\}_{n=1}^{\infty}$ are iid $\mathrm{Exp}(\lambda)$ rv's.

Pf. Partition an interval $(s,t]$ into finitely many intervals:



$= s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n =$

$J_i = (s_i, t_i]$

$K_i = (t_{i-1}, s_i]$

We will compute $\mathbb{P}(X_1 \in J_1, \ldots, X_n \in J_n)$

$$\{X_1 \in J_1, \ldots, X_n \in J_n\} = \{N(K_1) = \cdots = N(K_n) = 0\}$$
$$\cap \{N(J_1) = \cdots = N(J_{n-1}) = 1\}$$
$$\cap \{N(J_n) \geq 1\}$$

all independent!

$$\{X_1 \in J_1, \ldots, X_n \in J_n\}$$

$$= \{N(K_1) = \cdots = N(K_n) = 0\} \cap \{N(J_1) = \cdots = N(J_{n-1}) = 1\} \cap \{N(J_n) \geq 1\}$$

$$\mathbb{P}(N(J) = n) =$$

$$\mathbb{P}(N(J) \geq 1) =$$

$$\therefore \mathbb{P}(X_1 \in J_1, \ldots, X_n \in J_n) = \prod_{j=1}^{n} \qquad \prod_{j=1}^{n-1}$$

$$\therefore = \lambda^{n-1} \prod_{j=1}^{n-1} |J_j| \left(e^{-\lambda s_n} - e^{-\lambda t_n}\right)$$

Thus, $\mathbb{P}((X_1,\dots,X_n) \in J_1 \times \dots \times J_n) = \int_{J_1\times\dots\times J_n} \lambda^n e^{-\lambda x_n} \, dx_1 \dots dx_n$.

Now, $\sigma(\{J_1\times\dots\times J_n\}) = \mathcal{B}(\Delta_n)$

$\therefore \forall B \in \mathcal{B}(\Delta_n)$, $\mathbb{P}((X_1,\dots,X_n) \in B) = \int_B \lambda^n e^{-\lambda x_n} \, dx_1 \dots dx_n$.

It now follows from your HW (c.o.v.) that the $\xi_k = X_k - X_{k-1}$ are iid $\mathrm{Exp}(\lambda)$.

However, property 3 (that $t \mapsto N_t$ is right-continuous)
DOES NOT FOLLOW.

Eg. $M_t := \sum_{n=1}^{\infty} \mathbb{1}_{[0,t)}(\xi_n)$ has the same f.d. distributions,
but is not right-continuous.

Moral: path properties are outside the purview of the Markov transition kernels.