Given $\nu \in \mathrm{Prob}(S, \mathcal{B})$ and Markov transition operators $\{Q_{s,t}\}_{s \le t \in T}$, we constructed a Markov process as a measure on S^T:

$$X_t(\omega) = \omega(t)$$

$\mathbb{P}^\nu \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$

designed to have the right f.d. distributions.

Not every Markov process with this data $(\nu, Q_{s,t})$ "is" this one.

But every stochastic process can be identified with a measure on path space:

$$(X_t)_{t \in T} \rightsquigarrow P_X \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$$

$$P_X(E) = \mathbb{P}(\omega \in \Omega : (t \mapsto X_t(\omega))_{t \in T} \in E).$$

E a collection of paths $t \mapsto \alpha(t)$

P_X is the law of the process $t \mapsto X_t$.

Problem: $\mathcal{B}^{\otimes T} = \sigma\{\pi_t : S^T \to S\}_{t \in T}$ is <u>small</u>.

Eg. $C([0,\infty), \mathbb{R})$ is <u>not</u> in $\mathcal{B}(\mathbb{R})^{[0,\infty)}$

α continuous at <u>all</u> $t \in [0,\infty)$ ← uncountable.

Brownian Motion

An adapted stochastic process $(B_t)_{t\geq 0}$ is called a (standard) Brownian motion if

1. It has independent increments
2. $B_t - B_s \overset{d}{=} \mathcal{N}(0, t-s)$ for $0 \leq s < t < \infty$
3. $t \mapsto B_t$ is continuous.

} PhD thesis of Albert Einstein 1905.

We can construct a Markov process $(X_t)_{t\geq 0}$ satisfying (1-2) by Kolmogorov:

$$q_{s,t}(x, B) = \mathbb{E}\left[\mathbb{1}_B(x + B_t - B_s)\right]$$

$$= \int_B (2\pi(t-s))^{-1/2} e^{-(y-x)^2/2(t-s)} dy.$$

provided these satisfy Chapman-Kolmogorov.

$B_t - B_t = 0$: $q_{t,t}(x, B) = \mathbb{E}\left[\mathbb{1}_B(x)\right] = \delta_x(B)$

$$(Q_{s,t}f)(x) = \int_{\mathbb{R}} f(y)\, q_{s,t}(x,dy) = \int_{\mathbb{R}} f(y)\,(2\pi(t-s))^{-1/2} e^{-(x-y)^2/2(t-s)}\,dy$$

$$= (\gamma_{t-s} * f)(x).$$

$$\therefore\ Q_{r,s}(Q_{s,t}f) = Q_{r,s}(\gamma_{t-s} * f) = \gamma_{s-r} * (\gamma_{t-s} * f)$$

$$= (\gamma_{s-r} * \gamma_{t-s}) * f = \gamma_{r,t} * f = Q_{r,t}f. \ \checkmark$$

$Y \overset{d}{=} N(0, s-r)$ $\quad Z \overset{d}{=} N(0, t-s)$ indep.

$Y + Z \overset{d}{=} N(0, \cancel{s}-r+t-\cancel{s})$

Incorporating Continuity will take some major work.
One possibility: explicit construction of B_t, like we did with the Poisson process.

$\{\xi_n\}_{n=1}^{\infty}$ iid $N(0,1)$ random variables

$$B_t = \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\xi_n}{n} \sin\left(n\frac{\pi}{2}t\right)$$

Compare to Poisson process $(N_t)_{t\geq 0}$

We explicitly constructed it as

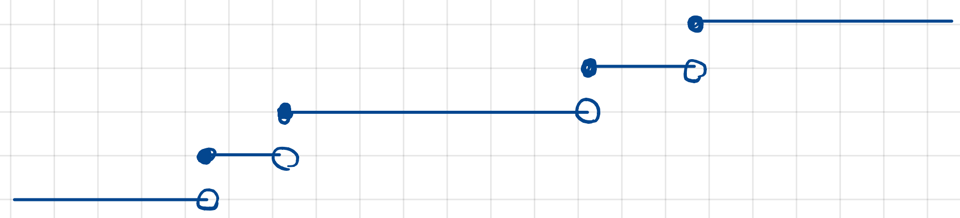
$$N_t(\omega) = \sup\{n \in \mathbb{N} : X_n \leq t\}, \text{ with } X_n = \xi_1 + \cdots + \xi_n$$

$\{\xi_k\}_{k=1}^{\infty}$ iid $\mathrm{Exp}(\lambda)$ rv's.

Properties: 1. Independent Increments

2. $N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s))$

3. $t \mapsto N_t$ is right-continuous, $\uparrow$



From 1-2:

$$q_{s,t}(n,B) = \mathbb{E}[\mathbb{1}_B(n + N_t - N_s)]$$

discrete, $q_{s,t}(n,B) = \sum_{m \in B} q_{s,t}(n,m)$

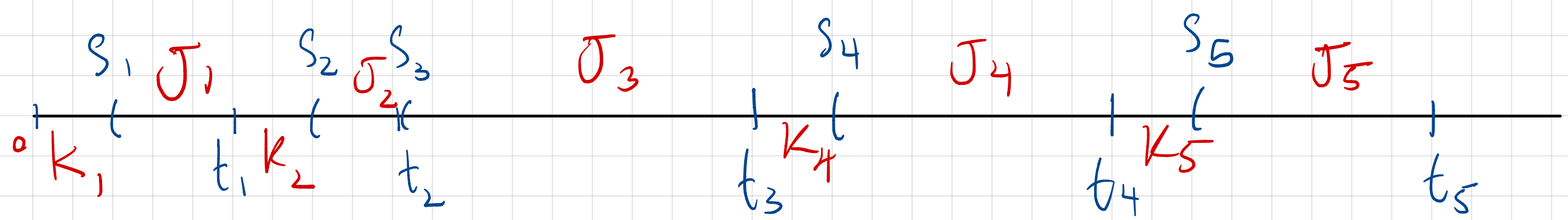
$$\begin{aligned} q_{s,t}(n,m) &= \mathbb{E}[\mathbb{1}_m(n + N_t - N_s)] \\ &= \mathbb{P}(n + N_t - N_s = m) \\ &= \mathrm{Poisson}(\lambda(t-s))\{m-n\} \end{aligned}$$

If $M_j \overset{d}{=} \mathrm{Poisson}(a_j)$ are independent, $M_1 + M_2 \overset{d}{=} \mathrm{Poisson}(a_1 + a_2)$

$\therefore$ Can verify C-K.

Prop: Let $(N_t)_{t\geq 0}$ be a process satisfying (1-2).
Set $X_n = \inf\{t \geq 0 : N_t = n\}$, and $\xi_n = X_n - X_{n-1}$, $n \geq 0$.
Then $\{\xi_n\}_{n=1}^{\infty}$ are iid $\text{Exp}(\lambda)$ rv's.

Pf. Partition an interval $(s,t]$ into finitely many intervals:



Set $N(s,t] := N_t - N_s$.

$= s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n =$

$J_i = (s_i, t_i]$

$K_i = (t_{i-1}, s_i]$

We will compute $\mathbb{P}(X_1 \in J_1, \dots, X_n \in J_n)$

$N_t - N_s \overset{d}{=} \text{Poisson}(\lambda(t-s)) \geq 0$.

$X_{n+1} \geq X_n$ a.s.

$$\{X_1 \in J_1, \dots, X_n \in J_n\} = \{N(K_1) = \cdots = N(K_n) = 0\}$$
$$\cap \{N(J_1) = \cdots = N(J_{n-1}) = 1\}$$
$$\cap \{N(J_n) \geq 1\}$$

all independent!

$\{X_1 \in J_1, \ldots, X_n \in J_n\}$

$= \{N(K_1) = \cdots = N(K_n) = 0\} \cap \{N(J_1) = \cdots = N(J_{n-1}) = 1\} \cap \{N(J_n) \geq 1\}$

$$\mathbb{P}(N(J) = n) = e^{-\lambda|J|} \frac{(\lambda|J|)^n}{n!}$$

$$\mathbb{P}(N(J) \geq 1) = 1 - \mathbb{P}(N(J) = 0) = 1 - e^{-\lambda|J|}$$

$$\therefore \mathbb{P}(X_1 \in J_1, \ldots, X_n \in J_n) = \prod_{j=1}^{n} e^{-\lambda|K_j|} \prod_{j=1}^{n-1} e^{-\lambda|J_j|} \lambda|J_j| \cdot (1 - e^{-\lambda|J_n|})$$

$$= \lambda^{n-1} \prod_{j=1}^{n-1} |J_j| \, e^{-\lambda|J_j|} e^{-\lambda|K_j|} \, e^{-\lambda|K_n|} (1 - e^{-\lambda|J_n|})$$

$$\overbrace{e^{-\lambda(t_j - \cancel{s_j} + \cancel{s_j} - t_{j-1})}}$$

$$= e^{-\lambda(t_{n-1} - 0)} \lambda^{n-1} \prod_{j=1}^{n-1} |J_j| \, e^{-\lambda(s_n - t_{n-1})} (1 - e^{-\lambda(t_n - s_n)})$$

$$\therefore = \lambda^{n-1} \prod_{j=1}^{n-1} |J_j| \underbrace{(e^{-\lambda s_n} - e^{-\lambda t_n})}_{\int_{J_n} \lambda e^{-\lambda x_n} dx_n} = \lambda^n \int_{J_1 \times \cdots \times J_n} e^{-\lambda x_n} dx_1 \cdots dx_n$$

Thus, $\mathbb{P}((X_1, \dots, X_n) \in J_1 \times \dots \times J_n) = \int_{J_1 \times \dots \times J_n} \lambda^n e^{-\lambda x_n} \, dx_1 \dots dx_n$.

Now, $\sigma(\{J_1 \times \dots \times J_n\}) = \mathcal{B}(\Delta_n)$

$\therefore \ \forall B \in \mathcal{B}(\Delta_n)$, $\mathbb{P}((X_1, \dots, X_n) \in B) = \int_B \lambda^n e^{-\lambda x_n} \, dx_1 \dots dx_n$.

It now follows from your HW (c.o.v.) that the $\xi_k = X_k - X_{k-1}$ are iid $\mathrm{Exp}(\lambda)$.

///

However, property 3 (that $t \mapsto N_t$ is right-continuous)

DOES NOT FOLLOW.

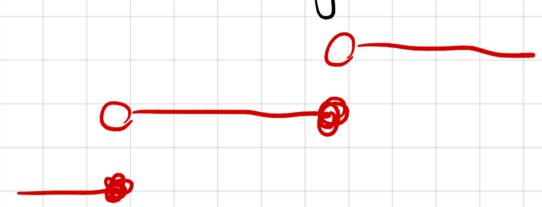
Eg. $M_t := \sum_{n=1}^{\infty} \mathbb{1}_{[0,t)}(\xi_n)$ has the same f.d. distributions,

$(0,t]$

but is not right-continuous.

$\{M_t = n\} = \{X_n < t \leq X_{n+1}\}$

$\leq \quad <$



<u>Moral</u>: path properties are outside the purview of the Markov transition kernels.