A Markov process (taking values in a regular Borel space $(S, \mathcal{B})$) comes with transition operators $Q_{s,t}$ on $\mathbb{B}(S, \mathcal{B})$, satisfying the Chapman-Kolmogorov equations:

This is analogous to saying: a sequence of i.i.d random variables comes with a sequence of joint laws $\mu_n \in \mathrm{Prob}(S^n, \mathcal{B}^{\otimes n})$ satisfying

$$\mu_n \otimes \mu_m = \mu_{n+m}$$

In [Lectures 16.1, 16.2] we considered the reverse question: given "consistent" measures μ_n can we actually find a sequence of i.i.d random variables where the joint law of the first n is μ_n?

The answer was yes, and we constructed the i.i.d variables on the probability space $S^{\mathbb{N}}$ via Kolmogorov's Extension Theorem.

Constructing Markov Processes

Let $(S,\mathcal{B})$ be a standard Borel space

Suppose $\nu \in \mathrm{Prob}(S,\mathcal{B})$, and $\{Q_{s,t}\}_{s\le t\in T}$ are Markov transition operators (i.e., probability kernels over $(S,\mathcal{B})^2$ satisfying the Chapman-Kolmogorov eq's).

We're going to construct a Markov process $(X_t)_{t\in T}$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ s.t. $\mathrm{Law}_{\mathbb{P}}(X_0)=\nu$ and the transition operators of $X_\cdot$ are the given $\{Q_{s,t}\}$.

We'll do this by taking $\Omega = S^T$

We construct X_t as the coordinate process

$$X_t(\omega) =$$

Thus, we need to define a σ-field on S^T and a probability measure $\mathbb{P}$ on that σ-field s.t. the coordinate process has the right initial distribution and transition operators.

Let $(S, \mathcal{B})$ be a measurable space, and T any set.

$S^T = \{\omega : T \to S\}$ is the product.

For $t \in T$, the projection $\pi_t : S^T \to S$ is the map

Define the product σ-field $\mathcal{B}^{\otimes T} :=$

We want to construct a measure $\mathbb{P} : \mathcal{B}^{\otimes T} \to [0,1]$ from information about its

finite-dimensional marginals:

For $\Lambda \subseteq T$ finite, let $\mu_\Lambda \in \mathrm{Prob}(S^\Lambda, \mathcal{B}^{\otimes \Lambda})$.

Want $\mathbb{P} \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$ s.t. $\pi_\Lambda^* \mathbb{P} = \mu_\Lambda$

What if $\Lambda, \Lambda' \subseteq T$ overlap?

There must be some consistency.

Need: $\Lambda' \subseteq \Lambda \implies \pi_{\Lambda'}^* \mu_\Lambda = \mu_{\Lambda'}$

Theorem: (Kolmogorov's (Extended) Extension Theorem)

Let $(S,\mathcal{B})$ be a standard Borel space, T any set.

For each finite subset $\Lambda \subseteq T$, let $\mu_\Lambda \in \mathrm{Prob}(S^\Lambda, \mathcal{B}^{\otimes\Lambda})$, and suppose

$$\forall \Lambda' \subseteq \Lambda, \quad \pi^*_{\Lambda'} \mu_\Lambda = \mu_{\Lambda'}.$$

Then $\exists!\ \mathbb{P} \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$ s.t. $\pi^*_\Lambda \mathbb{P} = \mu_\Lambda \quad \forall \Lambda \subseteq T$ finite.

Pf. Define an algebra $\mathcal{A} \subseteq \mathcal{B}^{\otimes T}$ by

$$\mathcal{A} = \bigcup_{\Lambda \subseteq T \text{ finite}} \sigma(\pi_\Lambda)$$

Define $\mathbb{P} : \mathcal{A} \to [0,1]$ by $\mathbb{P}(A) =$

This is well-defined:

$\pi_\Lambda^{-1}(B) = \pi_{\Lambda'}^{-1}(B') \Rightarrow \Lambda' \subseteq \Lambda, \; B = B' \times S^{\Lambda \setminus \Lambda'}$. $\therefore \mu_{\Lambda'}(B') = \pi_{\Lambda'}^{*}\mu_\Lambda(B')$

Thus $\mathbb{P}(\pi_\Lambda^{-1}(B)) := \mu_\Lambda(B)$ defines $\mathbb{P}$ on $\mathcal{A} = \bigcup_{\Lambda \subseteq T \text{ finite}} \sigma(\pi_\Lambda)$.

It's also a finitely additive measure: if $A_1, \dots, A_n \in \mathcal{A}$ disjoint

$$A_j = \pi_{\Lambda_j}^{-1}(B_j)$$

$$\therefore \mathbb{P}(A_1 \sqcup \cdots \sqcup A_n)$$

We've constructed a finitely-additive $\mathbb{P}$ on $\mathcal{A}$, and it (by design) has the "right" finite-dimensional marginals.

Now, all we have to do is show $\mathbb{P}$ is countably additive on $\mathcal{A}$.

Let $A_n \in \mathcal{A}$, $A_n \downarrow \emptyset$.

$\hookrightarrow A_n = \pi_{\Lambda_n}^{-1}(B_n)$ for some finite $\Lambda_n \subseteq T$, $B_n \in \mathcal{B}^{\otimes \Lambda_n}$

Set $L := \bigcup_n \Lambda_n$

Use the [Lecture 16.2] version of the Kolmogorov Extension Theorem!
The same consistency conditions implies

$$\exists ! \, \mathbb{P}_L \in \mathrm{Prob}(S^L, \mathcal{B}^{\otimes L}) \text{ s.t. } \pi_\Lambda^* \mathbb{P}_L = \mu_\Lambda \quad \forall \Lambda \subseteq L \text{ finite.}$$

Thus, $\mathbb{P}(A_n) = \mu_{\Lambda_n}(B_n)$

Theorem: Let $\nu \in \mathrm{Prob}(S, \mathcal{B})$ and let $\{Q_{s,t}\}_{s \le t \in T}$ be Markov transition operators on $(S, \mathcal{B})^2$. Then there exists a unique probability measure

$$\mathbb{P}^\nu \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$$

s.t. $X_t(\omega) = \omega(t)$ is a Markov process on $(S^T, \mathcal{B}^{\otimes T}, \mathbb{P}_\nu)$ with transition operators $\{Q_{s,t}\}_{s \le t \in T}$ and $X_0 \overset{d}{=} \nu$.

Pf. Idea: match the required f.d. distributions:

$$\mathrm{Law}_{\mathbb{P}^\nu}(X_{t_0}, X_{t_1}, \dots, X_{t_n})(dx_0 \cdots dx_{n+1}) = \nu(dx_0) \prod_{i=1}^{n} q_{t_{i-1}, t_i}(x_{i-1}, dx_i)$$

So define, for $\Lambda = \{0 = t_0 < t_1 < \dots < t_n\}$

$$\mu_\Lambda(dx_0 \cdots dx_n) =$$

We want to construct $\mathbb{P}^\nu$ s.t. $\pi_\Lambda{}^* \mathbb{P}^\nu = \mu_\Lambda$.

If so, $\mathbb{P}^\nu \{(X_{t_0}, \dots, X_{t_n}) \in B \in \mathcal{B}^{\otimes \Lambda}\}$

By Kolmogorov, we just need to show consistency of these μ_Λ.
To avoid a notational nightmare, let's just consider the example

$$\Lambda = \{0 = t_0 < t_1 < t_2\} \supseteq \Lambda' = \{0 = t_0 < t_2\}$$

$$\therefore \pi_{\Lambda'}^{*} \mu_\Lambda (B_0 \times B_2)$$

Following this calculation, we see that the Chapman-Kolmogorov equations imply consistency, giving us our measure $\mathbb{P}^\nu$ by Kolmogorov's Extension.

So, we have a process X_t with the right f.d. distributions, and $\therefore$ the right transition operators.

It remains to show X_t has the Markov property

Fix $0 = t_0 < t_1 < \cdots < t_{n-1} = s < t = t_n$. Let $h \in B(S^n, \mathcal{B}^{\otimes n})$, $g \in B(S, \mathcal{B})$.

$$\therefore \mathbb{E}^\nu [h(X_{t_0}, \ldots, X_{t_{n-1}}) g(X_{t_n})] = \int_{S^{n+1}} h(x_0, \ldots, x_{n-1}) g(x_n) \nu(dx_0) \prod_{i=1}^n q_{t_{i-1}, t_i}(x_{i-1}, dx_i)$$

That is: if $Y = h(X_{t_0}, \ldots, X_{t_{n-1}})$ for any $t_0 < t_1 < \cdots < t_{n-1} = s$, then

$$\mathbb{E}^\nu [g(X_t) Y] = \mathbb{E}^\nu [$$

By Dynkin, this $\therefore$ holds $\forall Y \in B(\Omega, \mathcal{F}_s^X)$.

$$\therefore \mathbb{E}[g(X_t) | \mathcal{F}_s^X] =$$