A Markov process (taking values in a regular Borel space $(S, \mathcal{B})$) comes with transition operators $Q_{s,t}$ on $\mathbb{B}(S,\mathcal{B})$, satisfying the Chapman-Kolmogorov equations:

$$Q_{r,t} = Q_{r,s} Q_{s,t} \qquad r \le s \le t$$

$$Q_{t,t} = I.$$

This is analogous to saying: a sequence of iid random variables comes with a sequence of joint laws $\mu_n \in \mathrm{Prob}(S^n, \mathcal{B}^{\otimes n})$ satisfying

$$\mu_n \otimes \mu_m = \mu_{n+m}$$

In [Lectures 16.1, 16.2] we considered the reverse question:
given "consistent" measures μ_n
can we actually find a sequence of iid random variables where the joint law of the first n is μ_n?

The answer was yes, and we constructed the iid variables on the probability space $S^{\mathbb{N}}$ via Kolmogorov's Extension Theorem.

Constructing Markov Processes

Let $(S,\mathcal{B})$ be a standard Borel space

Suppose $\nu \in \mathrm{Prob}(S,\mathcal{B})$, and $\{Q_{s,t}\}_{s\le t\in T}$ are Markov transition operators (i.e. probability kernels over $(S,\mathcal{B})^2$ satisfying the Chapman-Kolmogorov eq's).

We're going to construct a Markov process $(X_t)_{t\in T}$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ s.t. $\mathrm{Law}_{\mathbb{P}}(X_0)=\nu$ and the transition operators of X. are the given $\{Q_{s,t}\}$.

We'll do this by taking $\Omega = S^T = \{\omega : T\to S\}$

We construct X_t as the coordinate process

$$X_t(\omega) = \omega(t).$$

Thus, we need to define a σ-field on S^T and a probability measure $\mathbb{P}=\mathbb{P}^\nu$ on that σ-field s.t. the coordinate process has the right initial distribution and transition operators.

Let $(S,\mathcal{B})$ be a measurable space, and T any set.

$S^T = \{\omega : T \to S\}$ is the product.

For $t \in T$, the projection $\pi_t : S^T \to S$ is the map $\pi_t(\omega) = \omega(t)$.

Define the product σ-field $\mathcal{B}^{\otimes T} := \sigma(\pi_t : t \in T)$

$\pi_t : S^T \to (S,\mathcal{B})$

$\sigma(\pi_t) = \{\pi_t^{-1}(B) : B \in \mathcal{B}\}$

$= \{\omega \in S^T : \omega(t) \in B\}$

"cylinder sets"

We want to construct a measure $\mathbb{P} : \mathcal{B}^{\otimes T} \to [0,1]$ from information about its

finite-dimensional marginals:

For $\Lambda \subseteq T$ finite, let $\mu_\Lambda \in \mathrm{Prob}(S^\Lambda, \mathcal{B}^{\otimes \Lambda})$.

Want $\mathbb{P} \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$ s.t. $\pi_\Lambda^* \mathbb{P} = \mu_\Lambda$

What if $\Lambda, \Lambda' \subseteq T$ overlap?

There must be some consistency.

Need: $\Lambda' \subseteq \Lambda \Rightarrow \pi_{\Lambda'}^* \mu_\Lambda = \mu_{\Lambda'}$

$\uparrow$

"$(\pi_{\Lambda'}|_{S^\Lambda})^* \mu_\Lambda$"

Theorem: (Kolmogorov's (Extended) Extension Theorem)

Let $(S,\mathcal{B})$ be a standard Borel space, T any set.
For each finite subset $\Lambda \subseteq T$, let $\mu_\Lambda \in \mathrm{Prob}(S^\Lambda, \mathcal{B}^{\otimes\Lambda})$, and suppose

$$\forall \Lambda' \subseteq \Lambda, \quad \pi_{\Lambda'}^{*} \mu_\Lambda = \mu_{\Lambda'}.$$

Then $\exists!\, \mathbb{P} \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$ s.t. $\pi_\Lambda^{*} \mathbb{P} = \mu_\Lambda$ $\forall \Lambda \subseteq T$ finite.

Pf. Define an algebra $\mathcal{A} \subseteq \mathcal{B}^{\otimes T}$ by

union of algebras $\rightarrow$ $\mathcal{A} = \bigcup_{\Lambda \subseteq T \text{ finite}} \sigma(\pi_\Lambda)$ $\quad \{\pi_\Lambda^{-1}(B) : B \in \mathcal{B}^{\otimes\Lambda}\}$ $\quad \sigma(\mathcal{A}) = \mathcal{B}^{\otimes T}$

Define $\mathbb{P} : \mathcal{A} \to [0,1]$ by $\mathbb{P}(A) = \mu_\Lambda(B)$.
$A = \pi_\Lambda^{-1}(B)$ $\quad \Lambda \subseteq T$, $B \in \mathcal{B}^{\otimes\Lambda}$

This is well-defined: $\pi_\Lambda^{-1}(B) = \pi_{\Lambda'}^{-1}(B')$ $\quad (|\Lambda'| \le |\Lambda|)$

$\Lambda = \{t_1,\dots,t_n\}$ $\quad \pi_\Lambda^{-1}(B) = \{\omega : (\omega(t_1),\dots,\omega(t_n)) \in B\}$ $))$

$\Lambda' = \{s_1,\dots,s_m\}$ $\quad \pi_{\Lambda'}^{-1}(B') = \{\omega : (\omega(s_1),\dots,\omega(s_m)) \in B'\}$

$\Lambda' \subseteq \Lambda$, $\quad s_i = t_i$ $\quad i \le m$, $\quad B = B' \times S^{n-m}$

$\pi_\Lambda^{-1}(B) = \pi_{\Lambda'}^{-1}(B') \Rightarrow \Lambda' \subseteq \Lambda, \; B = B' \times S^{\Lambda \setminus \Lambda'}$. $\therefore \mu_{\Lambda'}(B') = \pi_{\Lambda'}^{*}\mu_\Lambda(B')$

$= \mu_\Lambda((\pi_{\Lambda'}|_\Lambda)^{-1}(B'))$

$= \mu_\Lambda(B' \times S^{\Lambda\setminus\Lambda'}) = \mu_\Lambda(B)$

Thus $\mathbb{P}(\pi_\Lambda^{-1}(B)) := \mu_\Lambda(B)$ defines $\mathbb{P}$ (well!)

on $\mathcal{A} = \bigcup_{\Lambda \subseteq T \text{ finite}} \sigma(\pi_\Lambda)$. Also $\mathbb{P}(S^T) = \mu_\Lambda(S^\Lambda) = 1$.

It's also a finitely additive measure: if $A_1, \dots, A_n \in \mathcal{A}$ disjoint

$$A_j = \pi_{\Lambda_j}^{-1}(\underset{\substack{\uparrow \\ \mathcal{B}^{\otimes \Lambda_j}}}{B_j}) = \pi_\Lambda^{-1}(\underset{\substack{\uparrow \\ \text{disjoint}}}{B_j'}) \quad \Lambda = \Lambda_1 \cup \dots \cup \Lambda_n, \quad B_j' = B_j \times S^{\Lambda \setminus \Lambda_j}$$

$$\therefore \mathbb{P}(A_1 \sqcup \dots \sqcup A_n) = \mu_\Lambda(B_1' \cup \dots \cup B_n') = \sum_{j=1}^n \mu_\Lambda(B_j') = \sum_{j=1}^n \mathbb{P}(A_j).$$

We've constructed a finitely-additive $\mathbb{P}$ on $\mathcal{A}$, and it (by design) has the "right" finite-dimensional marginals.

Now, all we have to do is show $\mathbb{P}$ is countably additive on $\mathcal{A}$.

Let $A_n \in \mathcal{A}$, $A_n \downarrow \emptyset$.

↳ $A_n = \pi_{\Lambda_n}^{-1}(B_n)$ for some finite $\Lambda_n \subseteq T$, $B_n \in \mathcal{B}^{\otimes \Lambda_n}$

Set $L := \bigcup_n \Lambda_n$ ⇐ countable.

Use the [Lecture 16.2] version of the Kolmogorov Extension Theorem!

The same consistency conditions implies

$$\exists! \; \mathbb{P}_L \in \mathrm{Prob}(S^L, \mathcal{B}^{\otimes L}) \text{ s.t. } \pi_\Lambda^* \mathbb{P}_L = \mu_\Lambda \quad \forall \Lambda \subseteq L \text{ finite.}$$

Thus, $\mathbb{P}(A_n) = \mu_{\Lambda_n}(B_n) = \pi^*_{\Lambda_n}\mathbb{P}_L(B_n) = \mathbb{P}_L(\tilde{B}_n) \to 0$.

$$= \mathbb{P}_L\big((\pi_{\Lambda_n}|_L)^{-1}(B_n)\big) \qquad ///$$

$$(\pi_{\Lambda_n}|_L)^{-1}(B_n) = B_n \times S^{L \setminus \Lambda_n} = \tilde{B}_n$$

$$A_n = \pi_{\Lambda_n}^{-1}\,(\pi_{\Lambda_n}|_L)(\tilde{B}_n)$$

$$\tilde{B}_n = (\pi_{\Lambda_n}|_L)^{-1}\big(\pi_{\Lambda_n}(A_n)\big) \downarrow \emptyset. \text{ b/c } A_n \downarrow \emptyset.$$

Theorem: Let $\nu \in \mathrm{Prob}(S,\mathcal{B})$ and let $\{Q_{s,t}\}_{s\le t\in T}$ be Markov transition operators on $(S,\mathcal{B})^2$. Then there exists a unique probability measure

$$\mathbb{P}^{\nu} \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T})$$

s.t. $X_t(\omega) = \omega(t)$ is a Markov process on $(S^T, \mathcal{B}^{\otimes T}, \mathbb{P}_\nu)$ with transition operators $\{Q_{s,t}\}_{s\le t\in T}$ and $X_0 \overset{d}{=} \nu$.

Pf. Idea: match the required f.d. distributions:

$$\mathrm{Law}_{\mathbb{P}^\nu}(X_{t_0}, X_{t_1}, \dots, X_{t_n})(dx_0 \cdots dx_{n+1}) = \nu(dx_0)\prod_{i=1}^{n} q_{t_{i-1},t_i}(x_{i-1}, dx_i)$$

So define, for $\Lambda = \{0 = t_0 < t_1 < \dots < t_n\}$

$$\mu_\Lambda(dx_0 \cdots dx_n) =$$

We want to construct $\mathbb{P}^\nu$ s.t. $\pi_\Lambda{}^{*}\mathbb{P}^\nu = \mu_\Lambda$.

If so, $\mathbb{P}^\nu\{(X_{t_0}, \dots, X_{t_n}) \in B \in \mathcal{B}^{\otimes\Lambda}\}$

$$= \mu_\Lambda\{\omega \in S^\Lambda : \underset{\substack{\shortparallel \\ (x_0, x_1, \dots, x_n)}}{(\omega(t_0), \dots, \omega(t_n))} \in B\}$$

By Kolmogorov, we just need to show consistency of these μ_Λ.

To avoid a notational nightmare, let's just consider the example

$$\Lambda = \{0 = t_0 < t_1 < t_2\} \supseteq \Lambda' = \{0 = t_0 < t_2\}$$

$$\therefore \ \pi_{\Lambda'}^{*} \mu_\Lambda (B_0 \times B_2) = \mu_\Lambda (B_0 \times S \times B_2) = \int_{B_0} \nu(dx_0) \int_S q_{t_0,t_1}(x_0, dx_1) \int_{B_2} q_{t_1,t_2}(x_1, dx_2)$$

$$= \int_{B_0 \times B_2} \nu(dx_0) \underbrace{\int_S q_{t_0,t_1}(x_0, dx_1) \underbrace{q_{t_1,t_2}(x_1,}_{} dx_2)}_{= q_{t_0,t_2}(x_0, dx_2) \ (C\text{-}K).}$$

$$\Big\downarrow$$

$$= \int_{B_0} \nu(dx_0) \int_{B_2} q_{t_0,t_2}(x_0, dx_2)$$

$$= \mu_{\Lambda'}(B_0 \times B_2)$$

Following this calculation, we see that the Chapman-Kolmogorov equations imply consistency, giving us our measure $\mathbb{P}^\nu$ by Kolmogorov's Extension.

So, we have a process X_t with the right f.d. distributions, and ∴ the right transition operators.

It remains to show X_t has the Markov property (wrt $(\mathcal{F}_t^X)_{t\in T}$)

Fix $0=t_0<t_1<\cdots<t_{n-1}=s<t=t_n$. Let $h\in B(S^n,\mathcal{B}^{\otimes n})$, $g\in B(S,\mathcal{B})$.

$$\therefore\ \mathbb{E}^\nu[h(X_{t_0},\dots,X_{t_{n-1}})g(X_{t_n})] = \int_{S^{n+1}} h(x_0,\dots,x_{n-1})\,g(x_n)\,\nu(dx_0)\prod_{i=1}^{n} q_{t_{i-1},t_i}(x_{i-1},dx_i)$$

$\int \cdot \, d\mathbb{P}^\nu$

$$= \int_{S^n} h(x_0,\dots,x_{n-1})(Q_{\underset{s}{t_{n-1}},\underset{t}{t_n}}g)(x_{n-1})\,\nu(dx_0)\prod_{i=1}^{n-1} q_{t_{i-1},t_i}(x_{i-1},dx_i)$$

$$= \mathbb{E}^\nu[h(X_{t_0},\dots,X_{t_{n-1}})(Q_{s,t}g)(X_s)]$$

That is: if $Y=h(X_{t_0},\dots,X_{t_{n-1}})$ for any $t_0<t_1<\cdots<t_{n-1}=s$,

then $$\mathbb{E}^\nu[g(X_t)Y]=\mathbb{E}^\nu[(Q_{s,t}g)(X_s)Y].$$

By Dynkin, this ∴ holds $\forall Y\in B(\Omega,\mathcal{F}_s^X)$.

$$\therefore\ \mathbb{E}[g(X_t)|\mathcal{F}_s^X] = (Q_{s,t}g)(X_s)$$
$$= \mathbb{E}_{\sigma(X_s)}[Q_{s,t}g(X_s)]$$
$$= \mathbb{E}[g(X_t)|X_s].\quad ///$$