Markov Processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ be a filtered probability space.
Let $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ be an adapted process, satisfying the Markov property.

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \quad \text{a.s.} \quad \forall\ s<t \text{ in } T,\ f \in B(S,\mathcal{B}).$$

Let's suppose $(S, \mathcal{B})$ is a regular Borel space.
Then there exists a regular conditional distribution $q_{s,t}$ of $X_t \mid X_s$:

What can we say about these transition operators?
Suppose $r<s<t$ in T. Then

$$Q_{r,t}f(X_r) = \mathbb{E}[f(X_t) \mid X_r]$$

I.e. $\int Q_{r,t}f(x)\,\mu_{X_r}(dx) =$

Thus $\{q_{r,t}\}_{r<t}$ satisfy

$$q_{r,t}(x,B) = \int q_{r,s}(x,dy)\, q_{s,t}(y,B) \text{ for } \mu_{X_r}\text{-a.e. } x.$$

To avoid possible measure-theoretic pitfalls, we strengthen this to hold $\forall x$.

[This could be problematic. What if $\mu_{X_r} \perp \mu_{X_s}$ for $s \neq r$?

Def: A collection of probability kernels $\{q_{s,t}\}_{s\leq t}$ (operators $\{Q_{s,t}\}_{s\leq t}$) Markov transition kernels (operators) if

1. $q_{s,s}(x,\cdot) = \delta_x$ $(Q_{s,s} = Id)$ $\forall s \in T,\ x \in S$.
2. $Q_{r,t} = Q_{r,s} Q_{s,t}$ $\forall\, r \leq s \leq t$

I.e. $q_{r,t}(x,B) = \int q_{r,s}(x,dy)\, q_{s,t}(y,B)$ for all $x \in S$.

1-2 are the Chapman-Kolmogorov equations.

Def: Given $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \in T}, \mathbb{P})$ and $(S, \mathcal{B})$, an adapted stochastic process $X_t : \Omega \to S$ is a Markov process if it satisfies the Markov property, and there exist Markov transition operators $\{Q_{s,t}\}_{s \leq t}$

$$\text{s.t.} \quad \mathbb{E}[f(X_t) | X_s] = Q_{s,t}(X_s) \quad \forall s \leq t \text{ in } T, \ f \in B(S, \mathcal{B}).$$

We've seen that the Markov property implies the Chapman-Kolmogorov equations. The converse is false.

Def: A process $(X_t)_{t \in T}$ has independent increments if, for all $t_0 < t_1 < \cdots < t_n$ in T,

$$X_{t_1} - X_{t_0}, \ X_{t_2} - X_{t_1}, \ldots, \ X_{t_n} - X_{t_{n-1}}$$

are independent.

Prop: An independent increment process X is a Markov process (wrt $\{\mathcal{F}_t^X\}_{t \in T}$), and $X_t - X_s$ is independent from $\mathcal{F}_s$ $\forall s < t$. The transition kernels are

$$q_{s,t}(x, B) =$$

Let's focus on discrete time, for a moment, wlog $T = \mathbb{N}$.

$$(X_n)_{n \in \mathbb{N}}, \quad \{Q_{m,n} : 0 \le m \le n < \infty\}$$

The Chapman-Kolmogorov equations imply that

$$Q_{m,n} =$$

$\therefore$ In this case it suffices to know the 1-step transition operators $\{Q_{m,m+1} : m \in \mathbb{N}\}$.

What if the state space is also discrete? I.e. S is countable.

$$\therefore \quad q_{m,m+1}(x, B) = \sum_{y \in B} q_{m,m+1}(x, y)$$

$\therefore$ C-K says

$$q_{m,n}(x,y) = \sum_{x_{m+1}, \dots, x_{n-1} \in S} q_{m,m+1}(x, x_{m+1})\, q_{m+1,m+2}(x_{m+1}, x_{m+2}) \cdots q_{n-1,n}(x_{n-1}, y)$$

Finite-Dimensional Distributions

Let $(X_t)_{t\in T}$ be a Markov process with transition kernels $\{q_{s,t}\}_{s\le t}$.
Fix any times $t_0 < t_1 < \cdots < t_n$. Let $\nu_0 = \mathrm{Law}(X_{t_0})$.

Prop: The law of $(X_{t_0}, X_{t_1}, \dots, X_{t_n})$ is

$$\mathrm{Law}(X_{t_0}, X_{t_1}, \dots, X_{t_n})(dx_0\, dx_1 \cdots dx_n) = \nu_0(dx_0)\, q_{t_0,t_1}(x_0, dx_1)\, q_{t_1,t_2}(x_1, dx_2) \cdots q_{t_{n-1},t_n}(x_{n-1}, dx_n)$$

Pf. By Dynkin, suffices to prove
Now proceed by induction.

$$E[f_0(X_{t_0}) f_1(X_{t_1}) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})]$$