Markov Processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ be a filtered probability space.

Let $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ be an adapted process, satisfying the Markov property.

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \quad \text{a.s.} \quad \forall\, s<t \text{ in } T,\ f \in B(S,\mathcal{B}).$$

Let's suppose $(S, \mathcal{B})$ is a regular Borel space.

Then there exists a regular conditional distribution $q_{s,t}$ of $X_t \mid X_s$:

$$\mathbb{E}[f(X_t) \mid X_s] = \int_S f(y)\, q_{s,t}(X_s, dy) =: Q_{s,t} f(X_s)$$

What can we say about these transition operators?

Suppose $r<s<t$ in T. Then $\sigma(X_r) \subseteq \mathcal{F}_r \subseteq \mathcal{F}_s$

$$Q_{r,t} f(X_r) = \mathbb{E}[f(X_t) \mid X_r] = \mathbb{E}[\mathbb{E}_{\mathcal{F}_s}[f(X_t)] \mid X_r]$$

$$= \mathbb{E}[Q_{s,t} f(X_s) \mid X_r] = Q_{r,s}(Q_{s,t} f)(X_r)$$

I.e. $\displaystyle\int Q_{r,t} f(x)\, \mu_{X_r}(dx) = \int (Q_{r,s} Q_{s,t} f)(x)\, \mu_{X_r}(dx) \quad \forall f$

$$\overset{\shortparallel}{\int f\, d(\mu_{X_r} \otimes q_{r,t})} \qquad \overset{\shortparallel}{\int f\, d(\mu_{X_r} \otimes (q_{r,s} \otimes q_{s,t}))}.$$

Thus $\{q_{r,t}\}_{r<t}$ satisfy

$$q_{r,t}(x,B) = \int q_{r,s}(x,dy)\, q_{s,t}(y,B) \text{ for } \mu_{X_r}\text{-a.e. } x.$$

I.e. $\forall f \in \mathbb{B}(S,\mathcal{B})$, $Q_{r,t} f = Q_{r,s} Q_{s,t} f$ $[\mu_{X_r}]$-a.s.

To avoid possible measure-theoretic pitfalls, we strengthen this to hold $\forall x$.

[This could be problematic. What if $\mu_{X_r} \perp \mu_{X_s}$ for $s \neq r$?

Def: A collection of probability kernels $\{q_{s,t}\}_{s\leq t}$ (operators $\{Q_{s,t}\}_{s\leq t}$) Markov transition kernels (operators) if

1. $q_{s,s}(x,\cdot) = \delta_x$ $(Q_{s,s} = Id)$ $\forall s \in T$, $x \in S$.
2. $Q_{r,t} = Q_{r,s} Q_{s,t}$ $\forall r \leq s \leq t$

   I.e. $q_{r,t}(x,B) = \int q_{r,s}(x,dy)\, q_{s,t}(y,B)$ for all $x \in S$.

1-2 are the Chapman-Kolmogorov equations.

Def: Given $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in T}, \mathbb{P})$ and $(S, \mathcal{B})$, an adapted stochastic process $X_t : \Omega \to S$ is a **Markov process** if it satisfies the Markov property, and there exist Markov transition operators $\{Q_{s,t}\}_{s\le t}$

$$\text{s.t.} \quad \mathbb{E}[f(X_t) \mid X_s] = Q_{s,t}(X_s) \quad \forall s\le t \text{ in } T,\ f\in B(S,\mathcal{B}).$$

We've seen that the Markov property implies (a.s. form) the Chapman-Kolmogorov equations. The converse is false. ← subtle.

Def: A process $(X_t)_{t\in T}$ has **independent increments** if, for all $t_0 < t_1 < \cdots < t_n$ in T,

$$X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_n} - X_{t_{n-1}}$$

are independent. E.g. random walks, Poisson process.

Prop: An independent increment process X is a Markov process (wrt $\{\mathcal{F}_t^X\}_{t\in T}$), and $X_t - X_s$ is independent from $\mathcal{F}_s$ $\forall s<t$. The transition kernels are

$$q_{s,t}(x,B) = \mathbb{E}[\mathbb{1}_B(x + X_t - X_s)]$$

Let's focus on discrete time, for a moment, wlog $T = \mathbb{N}$.

$$(X_n)_{n\in\mathbb{N}}, \quad \{Q_{m,n} : 0 \le m \le n < \infty\}$$

The Chapman-Kolmogorov equations imply that

$$Q_{m,n} = Q_{m,m+1} Q_{m+1,m+2} \cdots Q_{n-1,n}.$$

$\therefore$ In this case it suffices to know the 1-step transition operators $\{Q_{m,m+1} : m \in \mathbb{N}\}$.

What if the state space is also discrete? I.e. S is countable.

$$\therefore \quad q_{m,m+1}(x,B) = \sum_{y\in B} q_{m,m+1}(x,y)$$

This is a RCD of $X_{m+1} | X_m$

"path integrals"

$$q_{m,m+1}(x,y) = \begin{cases} \mathbb{P}(X_{m+1} = y \mid X_m = x) \\ 0 \quad \text{if } \mathbb{P}(X_m = x) = 0. \end{cases}$$

$\therefore$ C-K says

$$q_{m,n}(x,y) = \sum_{x_{m+1},\dots,x_{n-1}\in S} q_{m,m+1}(x,x_{m+1})\, q_{m+1,m+2}(x_{m+1},x_{m+2}) \cdots q_{n-1,n}(x_{n-1},y)$$

Finite-Dimensional Distributions

Let $(X_t)_{t\in T}$ be a Markov process with transition kernels $\{q_{s,t}\}_{s\le t}$.
Fix any times $t_0 < t_1 < \cdots < t_n$. Let $\nu_0 = \mathrm{Law}(X_{t_0})$.

Prop: The law of $(X_{t_0}, X_{t_1}, \dots, X_{t_n})$ is

$$\mathrm{Law}(X_{t_0}, X_{t_1}, \dots, X_{t_n})(dx_0\, dx_1 \cdots dx_n) = \nu_0(dx_0)\, q_{t_0,t_1}(x_0, dx_1)\, q_{t_1,t_2}(x_1, dx_2) \cdots q_{t_{n-1},t_n}(x_{n-1}, dx_n)$$

$$\mathbb{E}[f_0(X_{t_0}) f_1(X_{t_1}) \cdots f_n(X_{t_n})] = \mathbb{E}_{\nu_0}[f_0 Q_{t_0,t_1}(f_1 Q_{t_1,t_2}(\cdots(f_{n-1} Q_{t_{n-1},t_n} f_n)\cdots))].$$

Pf. By Dynkin, suffices to prove

Now proceed by induction. (base case: $\nu_0 = \mathrm{Law}(X_{t_0})$) ✓

$$\mathbb{E}\big[\mathbb{E}[f_0(X_{t_0}) f_1(X_{t_1}) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}}) \mid \mathcal{F}_{t_n}]\big] \qquad [g_n = f_n \cdot Q_{t_n,t_{n+1}} f_{n+1}]$$

$$= \mathbb{E}[f_0(X_{t_0}) \cdots f_n(X_{t_n}) \underbrace{\mathbb{E}[f_{n+1}(X_{t_{n+1}}) \mid \mathcal{F}_{t_n}]}]$$

$$\mathbb{E}[f_{n+1}(X_{t_{n+1}}) \mid X_{t_n}] = Q_{t_n,t_{n+1}} f_{n+1}(X_{t_n})$$

$$= \mathbb{E}[f_0(X_{t_0}) \cdots f_{n-1}(X_{t_{n-1}})\, g_n(X_{t_n})]$$

$$= \mathbb{E}_{\nu_0}[f_0 Q_{t_0,t_1}(f_1 Q_{t_1,t_2}(\cdots(f_{n-1} Q_{t_{n-1},t_n} \tilde{g}_n)\cdots))]$$

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