The Markov Property

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ be a filtered probability space, (with $T = \mathbb{N}$ or $T = \mathbb{R}$).

An adapted stochastic process $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ satisfies the Markov property if, for $f \in B(S, \mathcal{B})$,

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \text{ a.s. for all } s < t \text{ in } T.$$

This is equivalent to

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = F(X_s) \text{ a.s. for some } F \in B(S, \mathcal{B})$$

$(\Rightarrow)$

$(\Leftarrow)$ If $\mathbb{E}_{\mathcal{F}_s}[f(X_t)] = F(X_s)$

Let $\mathcal{F}_t^X := \sigma(X_s : s \le t)$.

Earlier, we derived a Markov property

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s^X] = \mathbb{E}[f(X_t) \mid X_s] \quad \forall f \in \mathbb{B}(S, \mathcal{B}),\ s<t.$$

This is implied by the $(\mathcal{F}_t)_{t\in T}$-defined Markov property.

Pf.

Note: if T is countable, it suffices to state the Markov property in the form $\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[f(X_{n+1}) \mid X_n]$

Induction: Suppose $\mathbb{E}[f(X_n) \mid \mathcal{F}_k] = \mathbb{E}[f(X_n) \mid X_k]$ for some $n \ge k$.

The Markov property is about the present vs. the past.
But it also tells us about the future.

Def: Given a stochastic process $(X_t)_{t\in T}$, the **future σ-field** $\mathcal{F}^X_{\geq s}$ is defined as

$$\mathcal{F}^X_{\geq s} := \sigma(X_t : t \geq s)$$

(It is a reverse filtration: if $s_1 \leq s_2$, $\mathcal{F}^X_{s_1} \supseteq \mathcal{F}^X_{s_2}$.)

Prop: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $(X_t)_{t\in T}$ be an adapted stochastic process satisfying the Markov property. Then for $s\in T$,

$$\mathbb{E}[Y \mid \mathcal{F}_s] = \mathbb{E}[Y \mid X_s] \qquad \forall Y \in B(\Omega, \mathcal{F}^X_{\geq s}).$$

Lemma: Let $\mathbb{M} = \{ g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_n(X_{t_n}) :$
$n \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_n,\ g_j \in \mathbb{B}(S, \mathcal{B}) \}$.

Then $\mathbb{M}$ is a multiplicative system, and $\sigma(\mathbb{M}) = \mathcal{F}^X_{\geq s}$.

Pf. Multiplicative system

If $A \in \mathcal{F}^X_{\geq s}$

If $A \in \sigma(\mathbb{M})$

Lemma: Let $\mathbb{H} = \{ Y \in \mathbb{B}(\Omega, \mathcal{F}^X_{\geq s}) : \mathbb{E}[Y | \mathcal{F}_s] = \mathbb{E}[Y | X_s] \}$.

Then $\mathbb{H}$ is a subspace, contains 1, and is closed under bounded convergence.

Pf. Subspace:

Contains 1 :

Closed under bounded convergence:

Prop: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $(X_t)_{t\in T}$ be an adapted stochastic process satisfying the Markov property. Then for $s \in T$,

$$\mathbb{E}[Y \mid \mathcal{F}_s] = \mathbb{E}[Y \mid X_s] \qquad \forall Y \in \mathcal{B}(\Omega, \mathcal{F}^X_{\geq s}).$$

Pf. WTS $\mathbb{H} = \mathcal{B}(\Omega, \mathcal{F}^X_{\geq s})$. By the lemmas, suffices to show $\mathbb{M} \subseteq \mathbb{H}$, where

$$\mathbb{M} = \{ g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_n(X_{t_n}) : n \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_n,\ g_j \in \mathcal{B}(S, \mathcal{B}) \}.$$

$$\mathbb{E}_{\mathcal{F}_s}[Y] = \mathbb{E}_{\mathcal{F}_s}[\mathbb{E}_{\mathcal{F}_{t_{n-1}}}[Y]]$$

$$= \mathbb{E}_{\mathcal{F}_s}[g_0(X_{t_0}) \cdots g_{n-1}(X_{t_{n-1}}))$$

$$= \mathbb{E}_{\mathcal{F}_s}[F(X_{t_0})]$$

Conditional Independence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space; $\mathcal{A}, \mathcal{B}, \mathcal{G} \subseteq \mathcal{F}$ sub-σ-fields.

Say that $\mathcal{A}, \mathcal{B}$ are **conditionally independent** given $\mathcal{G}$ if

$$\mathbb{P}(A \cap B \mid \mathcal{G}) = \mathbb{P}(A \mid \mathcal{G})\,\mathbb{P}(B \mid \mathcal{G}) \text{ a.s. } \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Equivalently: $\mathbb{E}[XY \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{G}]\,\mathbb{E}[Y \mid \mathcal{G}]$ a.s. $\forall X \in B(\Omega, \mathcal{A}), Y \in B(\Omega, \mathcal{B})$

Eg. Follows that, if $C \in \mathcal{G}$, $\mathbb{P}(C) > 0$,

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$ [Why?]

Eg. Let X, Y be iid with law $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. Then $\mathbb{P}(X=Y) = \frac{1}{2}$.

$$\mathbb{E}[X \mid X=Y] = \mathbb{E}[Y \mid X=Y] = \frac{\mathbb{E}[X \mathbb{1}_{X=Y}]}{\mathbb{P}(X=Y)} = \frac{1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{3}{2}$$

$$\mathbb{E}[XY \mid X=Y] = \frac{\mathbb{E}[XY \mathbb{1}_{X=Y}]}{\mathbb{P}(X=Y)} = \frac{1 \cdot 1 \cdot \frac{1}{4} + 2 \cdot 2 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{5}{4}$$

This gives us a "poetic" way to rephrase the Markov property.

Theorem: The Markov property says:

"Conditioned on the present, the past and the future are independent."

More precisely: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ be an adapted stochastic process. Then $(X_t)_{t\in T}$ satisfies the Markov property iff, for each $s \in T$,

$\mathcal{F}_s$ and $\mathcal{F}^X_{\geq s}$ are conditionally independent given $\sigma(X_s)$.

Pf. ($\Rightarrow$) Let $Z \in B(\Omega, \mathcal{F}_s)$, $Y \in B(\Omega, \mathcal{F}^X_{\geq s})$.

$\mathbb{E}[ZY \mid \mathcal{F}_s]$

$\therefore \ \mathbb{E}_{\sigma(X_s)}\big[\,\mathbb{E}[ZY] \mid \mathcal{F}_s\big]$

($\Leftarrow$) [HW].