The Markov Property

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ be a filtered probability space, (with $T = \mathbb{N}$ or $T = \mathbb{R}$).

An adapted stochastic process $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ satisfies the Markov property if, for $f \in B(S,\mathcal{B})$,

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \text{ a.s. for all } s<t \text{ in } T.$$

This is equivalent to

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = F(X_s) \text{ a.s. for some } F \in B(S,\mathcal{B})$$

$(\Rightarrow)$ Doob-Dynkin ✓

$(\Leftarrow)$ If $\mathbb{E}_{\mathcal{F}_s}[f(X_t)] = F(X_s) = \mathbb{E}_{\sigma(X_s)}[F(X_s)] = \mathbb{E}_{\sigma(X_s)}[\mathbb{E}_{\mathcal{F}_s}[f(X_t)]]$

$$= \mathbb{E}_{\sigma(X_s)}[f(X_t)] = \mathbb{E}[f(X_t) \mid X_s].$$

✓

Let $\mathcal{F}_t^X := \sigma(X_s : s \le t)$. ($X_\cdot$ is adapted iff $\mathcal{F}_t^X \subseteq \mathcal{F}_t \ \forall t$)

Earlier, we derived a Markov property

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s^X] = \mathbb{E}[f(X_t) \mid X_s] \quad \forall f \in \mathbb{B}(S, \mathcal{B}),\ s<t.$$

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \quad " \qquad " \quad s<t.$$

This is implied by the $(\mathcal{F}_t)_{t\in T}$-defined Markov property.

Pf. $\mathbb{E}[f(X_t) \mid \mathcal{F}_s^X] = \mathbb{E}_{\mathcal{F}_s^X}[\mathbb{E}_{\mathcal{F}_s}[f(X_t)]] = \mathbb{E}_{\mathcal{F}_s^X}[\mathbb{E}_{\sigma(X_s)}[f(X_t)]] = \mathbb{E}_{\sigma(X_s)}[f(X_t)]$.

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Note: if T is countable, it suffices to state the Markov property in the form $\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[f(X_{n+1}) \mid X_n]$ base case $k=n$

Induction: Suppose $\mathbb{E}[f(X_n) \mid \mathcal{F}_k] = \mathbb{E}[f(X_n) \mid X_k]$ for some $n \ge k$.

$\mathbb{E}_{\mathcal{F}_n}[f(X_{n+1})] = F(X_n)$ $\therefore$ $\mathbb{E}_{\mathcal{F}_k}[f(X_{n+1})] = \mathbb{E}_{\mathcal{F}_k}[\mathbb{E}_{\mathcal{F}_n}[f(X_{n+1})]] = \mathbb{E}_{\mathcal{F}_k}[F(X_n)]$

$\therefore$ $= \mathbb{E}[F(X_n) \mid X_k] = \mathbb{E}_{\sigma(X_k)}[\mathbb{E}_{\mathcal{F}_n}[f(X_{n+1})]] = \mathbb{E}[f(X_{n+1}) \mid X_k]$.

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The Markov property is about the present vs. the past.
But it also tells us about the future.

Def: Given a stochastic process $(X_t)_{t\in T}$, the **future σ-field** $\mathcal{F}^X_{\geq s}$ is defined as

$$\mathcal{F}^X_{\geq s} := \sigma(X_t : t \geq s)$$

(It is a reverse filtration: if $s_1 \leq s_2$, $\mathcal{F}^X_{s_1} \supseteq \mathcal{F}^X_{s_2}$.)

Prop: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $(X_t)_{t\in T}$ be an adapted stochastic process satisfying the Markov property. Then for $s \in T$,

$$\mathbb{E}[Y \mid \mathcal{F}_s] = \mathbb{E}[Y \mid X_s] \qquad \forall Y \in \mathbb{B}(\Omega, \mathcal{F}^X_{\geq s}).$$

Eg. If $g_0, g_1, \dots, g_n \in \mathbb{B}(S, \mathcal{B})$, $s = t_0 < t_1 < \cdots < t_n$,

$$Y = g_0(X_{t_0})\, g_1(X_{t_1}) \cdots g_n(X_{t_n}) \in \mathbb{B}(\Omega, \mathcal{F}^X_{\geq s})$$

Lemma: Let $\mathbb{M} = \{ g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_n(X_{t_n}) :$
$n \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_n,\ g_j \in \mathbb{B}(S, \mathcal{B}) \}$.

Then $\mathbb{M}$ is a multiplicative system, and $\sigma(\mathbb{M}) = \mathcal{F}^X_{\geq s}$.

Pf. Multiplicative system ✓

If $A \in \mathcal{F}^X_{\geq s} = \sigma(X_t : t \geq s)$ $\quad A = X_t^{-1}(B),\ B \in \mathcal{B},\ t \geq s$.
$= (\mathbb{1}_B \circ X_t)^{-1}(1) \in \sigma(\mathbb{M})$.

If $A \in \sigma(\mathbb{M})$ $\therefore$ $A = Y^{-1}(E),\ Y \in \mathbb{M},\ E \in \mathcal{B}(\mathbb{R})$

function of $X_{t_0}, X_{t_1}, \dots, X_{t_n}$ $\quad t_0, \dots, t_n \geq s$. ///

Lemma: Let $\mathbb{H} = \{ Y \in \mathbb{B}(\Omega, \mathcal{F}^X_{\geq s}) : \mathbb{E}[Y | \mathcal{F}_s] = \mathbb{E}[Y | X_s] \}$.
Then $\mathbb{H}$ is a subspace, contains 1, and is closed under bounded convergence.

Pf. Subspace: linearity of $\mathbb{E}_{\mathcal{G}}$ $\quad \mathcal{G} = \mathcal{F}_s$ or $\mathcal{G} = \sigma(X_s)$

Contains 1: b/c $\mathbb{E}_{\mathcal{G}}[1] = 1$. ✓

Closed under bounded convergence: cDCT. ✓ ///

Prop: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $(X_t)_{t\in T}$ be an adapted stochastic process satisfying the Markov property. Then for $s \in T$,

$$\mathbb{E}[Y \mid \mathcal{F}_s] = \mathbb{E}[Y \mid X_s] \qquad \forall Y \in B(\Omega, \mathcal{F}^X_{\geq s}).$$

Pf. WTS $\mathbb{H} = B(\Omega, \mathcal{F}^X_{\geq s})$. By the lemmas, suffices to show $\mathbb{M} \subseteq \mathbb{H}$, where

$$\mathbb{M} = \{ g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_n(X_{t_n}) : n \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_n,\ g_j \in B(S, \mathcal{B}) \}.$$

$$\mathbb{E}_{\mathcal{F}_s}[\underbrace{Y}_{g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_n(X_{t_n})}] = \mathbb{E}_{\mathcal{F}_s}[\mathbb{E}_{\mathcal{F}_{t_{n-1}}}[Y]]$$

$$\mathbb{E}_{\mathcal{F}_{t_{n-1}}}[Y] = g_0(X_{t_0}) g_1(X_{t_1}) \cdots g_{n-1}(X_{t_{n-1}}) \overbrace{\mathbb{E}_{\mathcal{F}_{t_{n-1}}}[g_n(X_{t_n})]}^{h(X_{t_{n-1}})}$$

$$= \mathbb{E}_{\mathcal{F}_s}[g_0(X_{t_0}) \cdots \underbrace{g_{n-1}(X_{t_{n-1}}) h(X_{t_{n-1}})}_{\tilde{g}_{n-1}(X_{t_{n-1}})}]$$

repeat ...

$$= \mathbb{E}_{\mathcal{F}_s}[F(X_{t_0})] = F(X_s)$$

$$\therefore \mathbb{E}_{\mathcal{F}_s}[Y] = F(X_s) = \mathbb{E}[F(X_s) \mid X_s] = \mathbb{E}[Y \mid X_s].$$

Conditional Independence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space; $\mathcal{A}, \mathcal{B}, \mathcal{G} \subseteq \mathcal{F}$ sub-σ-fields.

Say that $\mathcal{A}, \mathcal{B}$ are conditionally independent given $\mathcal{G}$ if

$$\mathbb{P}(A \cap B \mid \mathcal{G}) = \mathbb{P}(A \mid \mathcal{G})\, \mathbb{P}(B \mid \mathcal{G}) \text{ a.s. } \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

$$\mathbb{E}[\mathbb{1}_A \mathbb{1}_B \mid \mathcal{G}]$$

Equivalently: $\mathbb{E}[XY \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{G}]\, \mathbb{E}[Y \mid \mathcal{G}]$ a.s. $\forall X \in \mathbb{B}(\Omega, \mathcal{A}), Y \in \mathbb{B}(\Omega, \mathcal{B})$

Eg. Follows that, if $C \in \mathcal{G}$, $\mathbb{P}(C) > 0$,

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\, \mathbb{P}(B \mid C) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

[Why?]

Eg. Let X, Y be iid with law $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. Then $\mathbb{P}(X=Y) = \frac{1}{2}$.

$$\mathbb{E}[X \mid X=Y] = \mathbb{E}[Y \mid X=Y] = \frac{\mathbb{E}[X \mathbb{1}_{X=Y}]}{\mathbb{P}(X=Y)} = \frac{1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{3}{2}$$

$$\mathbb{E}[XY \mid X=Y] = \frac{\mathbb{E}[XY \mathbb{1}_{X=Y}]}{\mathbb{P}(X=Y)} = \frac{1 \cdot 1 \cdot \frac{1}{4} + 2 \cdot 2 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{5}{4}$$

$$\frac{5}{4} \neq \frac{3}{2} \cdot \frac{3}{2}. \quad !$$

This gives us a "poetic" way to rephrase the Markov property.

Theorem: The Markov property says:

"Conditioned on the present, the past and the future are independent."

More precisely: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space, and let $X_t : (\Omega, \mathcal{F}_t) \to (S, \mathcal{B})$ be an adapted stochastic process. Then $(X_t)_{t\in T}$ satisfies the Markov property iff, for each $s \in T$,

$$\mathcal{F}_s \text{ and } \mathcal{F}^X_{\geq s} \text{ are conditionally independent given } \sigma(X_s).$$

$$\underset{\sigma(X_t : t \leq s)}{\mathcal{F}_s \supseteq} \qquad \underset{\sigma(X_t : t \geq s)}{\mathcal{F}^X_{\geq s} =}$$

Pf. ($\Rightarrow$) Let $Z \in B(\Omega, \mathcal{F}_s)$, $Y \in B(\Omega, \mathcal{F}^X_{\geq s})$.

$$\mathbb{E}[ZY \mid \mathcal{F}_s] = Z\,\mathbb{E}[Y \mid \mathcal{F}_s] = Z\,\mathbb{E}[Y \mid X_s]$$

$$\therefore \mathbb{E}_{\sigma(X_s)}\left[\mathbb{E}[ZY] \mid \mathcal{F}_s\right] = \mathbb{E}_{\sigma(X_s)}[Z\,\mathbb{E}_{\sigma(X_s)}[Y]]$$

$$= \mathbb{E}_{\sigma(X_s)}[Z] \cdot \mathbb{E}_{\sigma(X_s)}[Y].$$

$$\| $$

$$\mathbb{E}_{\sigma(X_s)}[ZY].$$

($\Leftarrow$) [HW].