Recall the Exponential Distribution of rate λ:

$\xi \overset{d}{=} \mathrm{Exp}(\lambda)$ iff $\mathbb{P}(\xi \geq t) = e^{-\lambda t} \wedge 1$.

It often models wait-times, for one reason: for $s,t \geq 0$,

$\mathbb{P}(\xi \geq t+s \mid \xi \geq s)$

Rearranging this, $\mathbb{P}(\xi \geq t+s) = \mathbb{P}(\xi \geq t)\,\mathbb{P}(\xi \geq s)$.

This property uniquely pins down $\xi \overset{d}{=} \mathrm{Exp}(\lambda)$ for some $\lambda \geq 0$.

Let $\{\xi_k\}_{k=1}^{\infty}$ be iid $\mathrm{Exp}(\lambda)$ rv's. Set $X_n = \sum_{k=1}^{n} \xi_k$.

Define $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{(0,t]}(X_n)$

$\{N_t = n\} =$

The renewal counting process $(N_t)_{t\geq 0}$ is called the standard Poisson process (of rate λ)

To understand the f.d. distributions of $(N_t)_{t\geq 0}$, it's useful to compute the f.d. distributions of $(X_n)_{n\geq 1}$.

Notice: $X_n > X_{n-1} > \cdots > X_1 > 0$ a.s.

So the state space of $(X_1, \ldots, X_n)$ is not all of $\mathbb{R}^n$.

Def: $\Delta_n(t) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1 < x_2 < \cdots < x_n < t\}$

$\Delta_n := \bigcup_{t>0} \Delta_n(t) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1 < x_2 < \cdots < x_n\}$

Lemma: $\mathrm{Leb}_{\mathbb{R}^n}(\Delta_n(t)) = t^n/n!$

Pf.

Lemma: If $g \in \mathbb{B}(\Delta_n, \mathcal{B}(\Delta_n))$, then

$$\mathbb{E}[g(X_1, \dots, X_n)] = \int_{\Delta_n} g(x_1, \dots, x_n) \lambda^n e^{-\lambda x_n} dx_1 \cdots dx_n .$$

Pf. [HW]

Cor: $N_t \overset{d}{=} \text{Poisson}(\lambda t)$.

Pf. Need to compute $\mathbb{P}(N_t = n)$ for each $n \in \mathbb{N}$.

$$\{N_t = n\} = \{X_n \leq t < X_{n+1}\}$$

$$\therefore \mathbb{P}(N_t = n) = \mathbb{P}(X_n \leq t < X_{n+1})$$

$$= \mathbb{E}[\mathbb{1}_{X_n \leq t < X_{n+1}}]$$

Prop: Let $n \in \mathbb{N}$, $t > 0$. Let $U_1, \dots, U_n$ be iid $\mathrm{Unif}[0,t]$ rv's.

Then for any symmetric function

$$f : [0,t]^n \to \mathbb{R}$$

$$\mathbb{E}[f(X_1, \dots, X_n) \mid N_t = n] = \mathbb{E}[f(U_1, \dots, U_n)].$$

(I.e. the distribution of $(X_1, \dots, X_n)$ is the same as $(U_1, \dots, U_n)$ reordered to be non-decreasing - the so-called order statistics).

Pf. First note, like in the corollary, that $\{N_t = n\} = \{X_n \le t < X_{n+1}\}$.

$$\therefore \quad \mathbb{E}[f(X_1, \dots, X_n) \mathbb{1}_{\{N_t = n\}}] = \mathbb{E}[f(X_1, \dots, X_n) \mathbb{1}_{\{X_n \le t < X_{n+1}\}}]$$

$$= \int_{\Delta_{n+1}} f(x_1, \dots, x_n) \mathbb{1}_{x_n \le t < x_{n+1}} \lambda^{n+1} e^{-\lambda x_{n+1}} \, dx_1 \cdots dx_{n+1}$$

$$= \int_{\Delta_n(t)} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n \int_t^\infty \lambda^{n+1} e^{-\lambda x_{n+1}} \, dx_{n+1}$$

Thus $\mathbb{E}[f(X_1, \dots, X_n) : N_t = n]$

$$= \lambda^n e^{-\lambda t} \int_{\Delta_n(t)} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

$$= \frac{\lambda^n}{n!} e^{-\lambda t} \sum_{\sigma \in S_n} \int_{\Delta_n(t)} f(x_{\sigma 1}, \dots, x_{\sigma n})\, dx_1 \cdots dx_n$$

$$= \frac{\lambda^n}{n!} e^{-\lambda t} \int_{[0,t]^n} f(u_1, \dots, u_n)\, du_1 \cdots du_n$$

Finite Dimensional Distributions of N_t

With our renewal process X_n with $\text{Exp}(\lambda)$ iid inter-arrival times, recall that

$$N_t = \sum_{n=0}^{\infty} \mathbb{1}_{(0,t]}(X_n)$$

It is convenient to generalize this to a "stochastic counting measure":

For $B \in \mathcal{B}(\mathbb{R})$, $N(B) := \sum_{n=1}^{\infty} \mathbb{1}_B(X_n)$

Theorem: If $\{B_1, \dots, B_k\}$ form a partition of $[0,t]$, then $N(B_1), \dots, N(B_k)$ are independent. If $\text{Leb}(B) =: |B| < \infty$, then $N(B) \stackrel{d}{=} \text{Poisson}(\lambda |B|)$.

In particular, applying this to an interval partition $0 < t_1 < \dots < t_n = t$, we have $N(0,t_1], N(t_1,t_2], \dots, N(t_{n-1},t]$ independent

and $N(s,t] \stackrel{d}{=} \text{Poisson}(\lambda(t-s))$.

Pf. We will compute the characteristic function of $(N(B_1), \dots, N(B_k))$. This involves the functions

$$e^{i\eta_j N(B_j)} = e^{i\eta_j \sum_{\ell=1}^{\infty} \mathbb{1}_{B_j}(X_\ell)}$$

On the event $\{N_t = n\} = \{X_n \le t < X_{n+1}\}$

$$\mathbb{E}\left[e^{i(\eta_1, \dots, \eta_k)\cdot(N(B_1), \dots, N(B_k))} \mid N_t = n \right]$$

$$= \mathbb{E}\left[\prod_{j=1}^{k} e^{i\eta_j \sum_{\ell=1}^{n} \mathbb{1}_{B_j}(X_\ell)} \mid N_t = n \right]$$

$$= \mathbb{E}\left[\prod_{j=1}^{k} \prod_{\ell=1}^{n} e^{i\eta_j \mathbb{1}_{B_j}(U_\ell)} \right]$$

$$= \prod_{\ell=1}^{n} \mathbb{E}\left[\prod_{j=1}^{k} e^{i\eta_j \mathbb{1}_{B_j}(U_\ell)} \right]$$

If $U \in B_j$, $e^{i\eta_j \mathbb{1}_{B_j}(U)} = e^{i\eta_j}$

The B_j partition $[0,t]$.

$\therefore \prod_{j=1}^{k} e^{i\eta_j \mathbb{1}_{B_j}(U)} = e^{i\eta_j}$ for the unique j s.t. $U \in B_j$

We've shown that

$$\mathbb{E}\left[e^{i(\eta_1, \dots, \eta_k)\cdot(N(B_1), \dots, N(B_k))} \mid N_t = n \right] = t^{-n} \left(\sum_{j=1}^{k} e^{i\eta_j} |B_j| \right)^n$$

$$\therefore \chi_{(N(B_1), \dots, N(B_k))}(\xi_1, \dots, \xi_n) = \sum_{n=0}^{\infty} \left(\quad \right) \mathbb{P}(N_t = n)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} e^{i\eta_j} |B_j| \right)^n t^{-n} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda \sum_{j=1}^{k} e^{i\eta_j} |B_j| \right)^n$$

Note: $\chi_{\text{Poisson}(\lambda)}(\eta) = e^{\lambda(e^{i\eta} - 1)}$