Recall the Exponential Distribution of rate λ:

$\xi \overset{d}{=} \mathrm{Exp}(\lambda)$ iff $\mathbb{P}(\xi \geq t) = e^{-\lambda t} \wedge 1$.

It often models wait-times, for one reason: for $s,t \geq 0$,

$$\mathbb{P}(\xi \geq t+s \mid \xi \geq s) = \frac{\mathbb{P}(\xi \geq t+s)}{\mathbb{P}(\xi \geq s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= \mathbb{P}(\xi \geq t)$$

Rearranging this, $\mathbb{P}(\xi \geq t+s) = \mathbb{P}(\xi \geq t)\,\mathbb{P}(\xi \geq s)$.

This property uniquely pins down $\xi \overset{d}{=} \mathrm{Exp}(\lambda)$ for some $\lambda \geq 0$.

$f(t) = \mathbb{P}(\xi \geq t)$, $f : [0,\infty) \to [0,1]$ $\quad f \downarrow \quad \therefore$ diff'ble [Leb]-a.e.

$f(t+s) = f(t)f(s)$

$$\frac{f(t+h) - f(t)}{h} = \frac{f(t)f(h) - f(t)}{h} = f(t)\frac{f(h)-1}{h}$$

$\therefore f'(0) = -\lambda$ exists, $\lambda \geq 0$.

$\therefore f'(t) = -\lambda f(t)$

$f(t) = \underbrace{f(0)}_{1}\, e^{-\lambda t}$

Let $\{\xi_k\}_{k=1}^{\infty}$ be iid $\mathrm{Exp}(\lambda)$ rv's. Set $X_n = \sum_{k=1}^{n} \xi_k$.

Define $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{(0,t]}(X_n) = \sup\{n : X_n \le t\}$

$\{N_t = n\} = \{X_n \le t < X_{n+1}\}$

The renewal counting process $(N_t)_{t\ge 0}$ is called the standard Poisson process (of rate λ)

To understand the f.d. distributions of $(N_t)_{t\ge 0}$, it's useful to compute the f.d. distributions of $(X_n)_{n\ge 1}$.

Notice: $X_n > X_{n-1} > \cdots > X_1 > 0$ a.s.

So the state space of $(X_1, \dots, X_n)$ is not all of $\mathbb{R}^n$.

Def: $\Delta_n(t) := \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1 < x_2 < \cdots < x_n < t\}$

$\Delta_n := \bigcup_{t>0} \Delta_n(t) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1 < x_2 < \cdots < x_n\}$

Lemma: $\mathrm{Leb}_{\mathbb{R}^n}(\Delta_n(t)) = t^n/n!$ $\quad [0,t]^n = \bigcup_{\sigma\in S_n} U_\sigma(\Delta_n(t)) \cup$ lower-dim pc's.

Pf. For $\sigma \in S_n$, $U_\sigma(x_1, \dots, x_n) = (x_{\sigma 1}, \dots, x_{\sigma n})$ $\quad \therefore \underbrace{\mathrm{Leb}([0,t]^n)}_{= t^n} = \sum_{\sigma\in S_n} \mathrm{Leb}[(\Delta_n(t))] = n!\,\mathrm{Leb}(\Delta_n(t))$

Lemma: If $g \in B(\Delta_n, \mathcal{B}(\Delta_n))$, then

$$E[g(X_1,\dots,X_n)] = \int_{\Delta_n} g(x_1,\dots,x_n)\,\lambda^n e^{-\lambda x_n}\,dx_1\cdots dx_n.$$

Pf. [HW]

Cor: $N_t \overset{d}{=} \text{Poisson}(\lambda t)$.

Pf. Need to compute $P(N_t = n)$ for each $n \in \mathbb{N}$.

$$\{N_t = n\} = \{X_n \le t < X_{n+1}\}$$

$$\therefore\ P(N_t = n) = P(X_n \le t < X_{n+1})$$

$$= E[\mathbb{1}_{X_n \le t < X_{n+1}}]$$

$$= \int_{\Delta_{n+1}} \mathbb{1}_{x_n \le t}\,\mathbb{1}_{t < x_{n+1}}\,\lambda^{n+1} e^{-\lambda x_{n+1}}\,dx_1\cdots dx_{n+1}$$

$$= \underbrace{\int_{\Delta_n(t)} dx_1\cdots dx_n}_{\frac{t^n}{n!}}\ \underbrace{\int_t^\infty \lambda^{n+1} e^{-\lambda x_{n+1}}\,dx_{n+1}}_{\lambda^n e^{-\lambda t}}$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \text{Poisson}\{n\}.$$

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Prop: Let $n \in \mathbb{N}$, $t > 0$. Let $U_1, \dots, U_n$ be iid $\mathrm{Unif}[0,t]$ rv's. Then for any symmetric function

$$f : [0,t]^n \to \mathbb{R}$$

$$\mathbb{E}[f(X_1, \dots, X_n) \mid N_t = n] = \mathbb{E}[f(U_1, \dots, U_n)].$$

(I.e. the distribution of $(X_1, \dots, X_n)$ is the same as $(U_1, \dots, U_n)$ reordered to be non-decreasing - the so-called order statistics).

Pf. First note, like in the corollary, that $\{N_t = n\} = \{X_n \le t < X_{n+1}\}$.

$$\therefore \mathbb{E}[f(X_1, \dots, X_n) \mathbb{1}_{\{N_t = n\}}] = \mathbb{E}[\underbrace{f(X_1, \dots, X_n) \mathbb{1}_{\{X_n \le t < X_{n+1}\}}}_{g(X_1, \dots, X_{n+1})}]$$

$$= \int_{\Delta_{n+1}} f(x_1, \dots, x_n) \mathbb{1}_{x_n \le t < x_{n+1}} \lambda^{n+1} e^{-\lambda x_{n+1}} \, dx_1 \cdots dx_{n+1}$$

$$= \int_{\Delta_n(t)} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n \underbrace{\int_t^\infty \lambda^{n+1} e^{-\lambda x_{n+1}} \, dx_{n+1}}_{\lambda^n e^{-\lambda t}}$$

Thus $\mathbb{E}[f(X_1, \dots, X_n) : N_t = n]$

$$= \lambda^n e^{-\lambda t} \int_{\Delta_n(t)} \underbrace{f(x_1, \dots, x_n)}_{\frac{1}{n!}\sum_{\sigma \in S_n} f(x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n})} dx_1 \cdots dx_n$$

$$= \frac{\lambda^n}{n!} e^{-\lambda t} \sum_{\sigma \in S_n} \int_{\Delta_n(t)} f(x_{\sigma 1}, \dots, x_{\sigma n}) \, dx_1 \cdots dx_n$$

c.o.v. $(x_{\sigma 1}, \dots, x_{\sigma n}) = (u_1, \dots, u_n)$

$\Delta_n(t) \to \Delta_n^\sigma(t)$

$$\sum_{\sigma \in S_n} \int_{\Delta_n^\sigma(t)} f(u_1, \dots, u_n) \, du_1 \cdots du_n .$$

$\bigcup_{\sigma \in S_n} \Delta_n^\sigma(t) = [0,t]^n \setminus$ lower-dim.

$\frac{1}{t^n}$

$$= \underbrace{\frac{(t\lambda)^n}{n!} e^{-\lambda t}}_{\mathbb{P}(N_t = n)} \int_{[0,t]^n} f(u_1, \dots, u_n) \, du_1 \cdots du_n$$

$\mathbb{E}[f(U_1, \dots, U_n)]$.

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Finite Dimensional Distributions of N_t

With our renewal process X_n with $\text{Exp}(\lambda)$ iid inter-arrival times, recall that

$$N_t = \sum_{n=0}^{\infty} \mathbb{1}_{(0,t]}(X_n)$$

It is convenient to generalize this to a "stochastic counting measure":

For $B \in \mathcal{B}(\mathbb{R})$, $N(B) := \sum_{n=1}^{\infty} \mathbb{1}_B(X_n)$ $\quad N_t = N(0,t]$.

Theorem: If $\{B_1, \dots, B_k\}$ form a partition of $[0,t]$, then $N(B_1), \dots, N(B_k)$ are independent. If $\text{Leb}(B) =: |B| < \infty$, then $N(B) \overset{d}{=} \text{Poisson}(\lambda |B|)$.

In particular, applying this to an interval partition $0 < t_1 < \dots < t_n = t$, we have

$$\underset{\substack{\shortparallel \\ N_{t_1}}}{N(0,t_1]}, \quad \underset{\substack{\shortparallel \\ N_{t_2} - N_{t_1}}}{N(t_1,t_2]}, \quad \dots, \quad \underset{\substack{\shortparallel \\ N_t - N_{t_{n-1}}}}{N(t_{n-1},t]} \quad \text{independent}$$

and $N(s,t] = N_t - N_s \overset{d}{=} \text{Poisson}(\lambda(t-s))$.

Pf. We will compute the characteristic function of $(N(B_1), \ldots, N(B_k))$. This involves the functions

$$e^{i\eta_j N(B_j)} = e^{i\eta_j \sum_{\ell=1}^{n} \mathbb{1}_{B_j}(X_\ell)}$$

On the event $\{N_t = n\} = \{X_n \le t < X_{n+1}\}$

$\mathbb{1}_{B_j}(X_\ell) \le \mathbb{1}_{[0,t]}(X_\ell) = 0$ for $\ell > n$.

$$\mathbb{E}\left[e^{i(\eta_1,\ldots,\eta_k)\cdot(N(B_1),\ldots,N(B_k))} \mid N_t = n\right]$$

$$= \mathbb{E}\Big[\prod_{j=1}^{k} e^{i\eta_j \underbrace{\sum_{\ell=1}^{n} \mathbb{1}_{B_j}(X_\ell)}_{\text{symmetric fn of } X_\ell\text{'s}}} \mid N_t = n\Big] = \mathbb{E}\Big[\prod_{j=1}^{k} e^{i\eta_j \sum_{\ell=1}^{n} \mathbb{1}_{B_j}(U_\ell)}\Big]$$

$U_1, \ldots, U_n$ iid $\mathrm{Unif}[0,t]$.

If $u \in B_j$, $e^{i\eta_j \mathbb{1}_{B_j}(u)} = e^{i\eta_j}$

The B_j partition $[0,t]$.

$\therefore \prod_{j=1}^{k} e^{i\eta_j \mathbb{1}_{B_j}(u)} = e^{i\eta_j}$ for the unique j s.t. $u \in B_j$

$$= \sum_{j=1}^{k} e^{i\eta_j} \mathbb{1}_{B_j}(u).$$

$$= \mathbb{E}\left[\prod_{j=1}^{k}\prod_{\ell=1}^{n} e^{i\eta_j \mathbb{1}_{B_j}(U_\ell)}\right]$$

$$= \prod_{\ell=1}^{n} \underbrace{\mathbb{E}\left[\prod_{j=1}^{k} e^{i\eta_j \mathbb{1}_{B_j}(U_\ell)}\right]}_{\sum_{j=1}^{k} e^{i\eta_j} P(U_\ell \in B_j)} = \left(\sum_{j=1}^{k} e^{i\eta_j} \frac{|B_j|}{t}\right)^{n}$$

$P(U_\ell \in B_j) = \dfrac{|B_j|}{t}$.

We've shown that

$$\mathbb{E}\left[e^{i(\eta_1,\dots,\eta_k)\cdot(N(B_1),\dots,N(B_k))} \mid N_t = n\right] = t^{-n}\left(\sum_{j=1}^{k} e^{i\eta_j}|B_j|\right)^n$$

$$\therefore \chi_{(N(B_1),\dots,N(B_k))}(\xi_1,\dots,\xi_n) = \sum_{n=0}^{\infty}\left(\quad\right)\mathbb{P}(N_t = n)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{k} e^{i\eta_j}|B_j|\right)^n t^{-n}\cdot\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

$$= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda\sum_{j=1}^{k} e^{i\eta_j}|B_j|\right)^n$$

$$= e^{-\lambda t} e^{\lambda \sum_{j=1}^{k} e^{i\eta_j}|B_j|} = \prod_{j=1}^{k} e^{\lambda |B_j|(e^{i\eta_j}-1)}$$

$$\lambda t = \lambda(|B_1| + \dots + |B_k|)$$

Note: $\chi_{\text{Poisson}(\lambda)}(\eta) = e^{\lambda(e^{i\eta}-1)}$.

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