Introduction to Stochastic Processes

$(\Omega, \mathcal{F}, \mathbb{P})$ $\qquad$ $(S, \mathcal{B})$

$T \leftarrow$ ordered set (with minimal element) $T = \mathbb{N}$, $T = [0, \infty)$, $[a, b]$

A stochastic process is a collection $\{X_t\}_{t \in T}$ of random variables $X_t : (\Omega, \mathcal{F}) \to (S, \mathcal{B})$.

The minimal information we'd like to have about a stochastic process is its finite-dimensional distributions:

For each finite set $\Lambda \subseteq T$, the measures $\mathrm{Law}_{\mathbb{P}}(X_t)_{t \in \Lambda} \in \mathrm{Prob}(S^\Lambda, \mathcal{B}^{\otimes \Lambda})$

usually inadequate to characterize $(X_t)_{t \in T}$.

Eg. $T = \mathbb{N}$, $\{\xi_k\}_{k=1}^{\infty}$ iid in $\mathbb{R}^d$. Set $X_n = \sum_{k=1}^{n} \xi_k$. $\qquad \xi_k \overset{d}{=} \mu$

Can explicitly describe the f.d. distributions.

$\hookrightarrow (X_1, X_3, X_4) = F(X_1, X_3 - X_1, X_4 - X_3) \overset{d}{=} F(\xi_1, \xi_2 + \xi_3, \xi_4) \leftarrow \mu \otimes \mu * \mu \otimes \mu.$

$F(x, y, z) = (x, x+y, x+y+z)$

As we saw in [Lecture 34.2], it is often important to understand $\sigma(X_1, X_2, \dots, X_n)$ as n varies.

Def: A collection $(\mathcal{F}_t)_{t\in T}$ of σ-fields is called a *filtration* if $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \le t$ in T.

E.g. If $\{X_n\}_{n=1}^{\infty}$ is a seq. of rvs, $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ is a filtration.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathcal{F}_t \subseteq \mathcal{F}$ $\forall t \in T$, then $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ is called a *filtered probability space*.

A stochastic process $(X_t)_{t\in T}$ (with $X_t : (\Omega, \mathcal{F}) \to (S, \mathcal{B})$) is called *adapted* if X_t is $\mathcal{F}_t / \mathcal{B}$-measurable $\forall t \in T$.

E.g. If we set $\mathcal{F}_t = \sigma(X_t)$ then X_t is clearly adapted.

typically not a filtration: $\sigma(X_s) \not\subseteq \sigma(X_t)$

$$\mathcal{F}_t = \sigma(\{X_s : s \le t\})$$

Usually we can safely take this as the filtration. But if we have more than one process around, we may need a more general filtration.

Eg. $\{\xi_k\}_{k=1}^{\infty}$ iid, $X_n = \xi_1 + \cdots + \xi_n$. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

$$= \sigma(\xi_1, \dots, \xi_n)$$

$$\mathbb{E}[g(X_n) \mid \mathcal{F}_k] = \mathbb{E}[g(X_k + \underbrace{\xi_{k+1} + \cdots + \xi_n}) \mid \sigma(X_1, \dots, X_k)]$$

indep.

$$= \mathbb{E}[g(x + \xi_{k+1} + \cdots + \xi_n)] \Big|_{x = X_k}.$$

$$= \mathbb{E}[g(X_k + \xi_{k+1} + \cdots + \xi_n) \mid X_k] = \mathbb{E}[g(X_n) \mid X_k].$$

This process satisfies the Markov property.

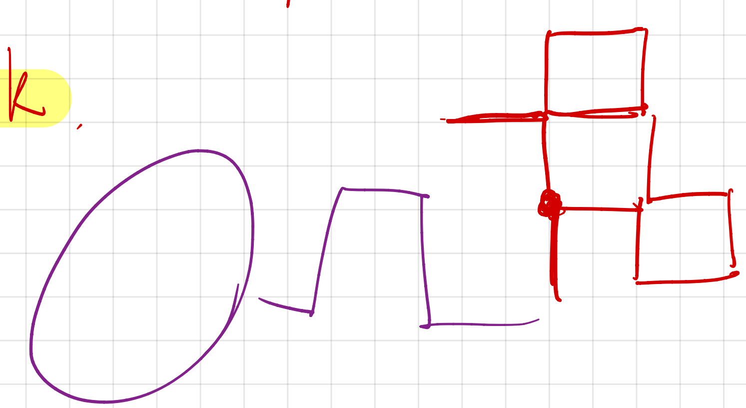
(Indeed, it fits the "random dynamics" model: $X_{n+1} = f(X_n, \xi_{n+1})$, $X_0 = 0$.)

$f(x,y) = x+y$.

In the special case (on $\mathbb{R}^d$) that the law of ξ_1 is

$$\mathbb{P}(\xi_1 = \pm e_j) = \frac{1}{2d} \qquad 1 \le j \le d \qquad \mathbb{R}^1: \ \xi_1 \overset{d}{=} \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1$$

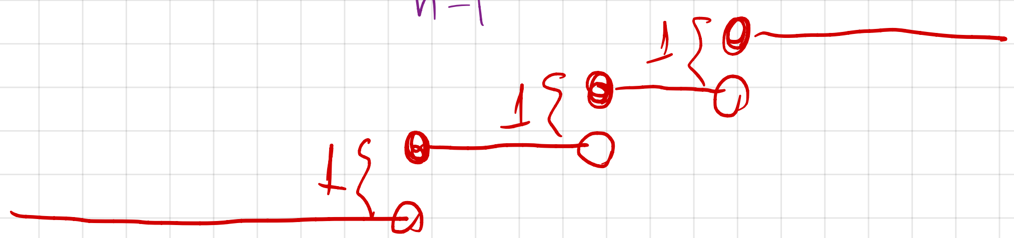
we call this stochastic process simple random walk.



E.g. Suppose $(X_n)_{n\geq 0}$ is a non-decreasing process

$$X_n = \sum_{k=1}^{n} \xi_k \qquad \xi_k \in [0,\infty)$$

Define $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{[0,t]}(X_n) = \sup\{n : X_n \leq t\}$



Any such $(X_n)_{n\geq 0}$ $\uparrow$ is called an **arrival process**.

The process $(N_t)_{t\geq 0}$ is the associated **counting process**.

If we set $\xi_n := X_n - X_{n-1}$, the $\{\xi_n\}_{n=1}^{\infty}$ are the **inter-arrival times**.

If the inter-arrival times are iid, $(X_n)_{n\geq 0}$ is a **renewal process**.

The most important example of (the counting process associated to) a renewal process is the **Poisson process**, which we'll study next time.